Utility and Beyond
a critical examination of certain established reasons given for learning mathematics

Thesis submitted in fulfilment of the degree of PhD from

University of London Institute of Education 1999

Peter John Huckstep

Abstract

In the Introduction I explore the reasons for my enquiry, and outline the inadequacies of some of the existing attempts to determine the *aims and purposes of mathematics* in education. In Part 1 I discuss the scope and validity of the justification of mathematics on the basis of its supposed *usefulness*. From there I defend the view that, in principle at least, there are different *kinds* of reasons for learning mathematics.

In Parts 2 and 3 I attempt to explain whether or not mathematics is *fit* for two particular non-utility purposes claimed by various writers, and if so, how. Thus, in Part 2, I examine the rather strong claim that mathematics is a *fine art*, and hence or otherwise that it is a source of *aesthetic* satisfaction. In Part 3, I explore the claim that mathematics provides *mental training*. Here I shall show that 'mental training', is a broad notion ranging from the rather moral character training to the more restricted notion of training in logic. Between these extremes lies a more modest notion which I argue is the most plausible.

The thesis is thus both a *history of ideas* and a *clarification of the concepts* used in describing fairly established purposes and rejecting those that seem to me to be unattainable or at least scarcely attainable by studying mathematics. The reason for the study is twofold. I see it as a particular case of the general enquiry into the aims of education. So that my conclusions should inform those who want to justify the place of mathematics on the curriculum. Also, however, I want to suggest that the purposes of mathematics education are internally related to understanding the subject, so that the *pupil* will gain *understanding* from a clearer notion of what he or she is doing mathematics for.

CONTENTS

Introduction

> It is thought that every activity, artistic or scientific, in fact every deliberate action or pursuit, has for its object the attainment of some good. We may therefore assent to the view which has been expressed that 'the good' is 'that at which all things aim'.
>
> (Aristotle, 1953 edn p.25)

It is a commonplace remark to say that mathematics is a difficult subject to learn. Even if there is no kudos attached to failure at mathematics at school, there is scarcely any shame in announcing that one was never very good at the subject, that one failed the exams once or twice or maybe more. Music, Foreign Languages, and Geography have their difficulties too. Indeed it would be quite possible to master mathematics and yet still find these latter subjects difficult. To be good at mathematics does not imply that any other subject will be learnt without difficulty. Yet mathematics is difficult in another way. It is difficult to learn and it seems it *must* be learnt. This creates panic. But one cannot, as it were, leave it on the side of one's plate. Even if there is no shame attached to failing at mathematics, to have failed at it is often regarded as a sign that one might not be 'academic' after all. It is one of the school subjects which traditionally puts one through one's paces intellectually. To make matters worse, many people are not clear why they learn it. It takes little to convince anyone that it has played an indispensable role in shaping modern civilisation. There seems hardly an area of life in which mathematics does not play a part. Moreover, most people will concede that there is *some* mathematics that we all need to know and understand in order to survive. But they cannot see the point of much of the mathematics which they struggled to learn.

This is not simply the case for those for whom one would have to invent hypothetical cases when they might need to use, say, simultaneous equations or even finding the number of sticks required to make the nth figure in a sequence of growing stick figures. Time and time again, in my professional life as a teacher educator, trainee primary teachers tell me that my primary curriculum mathematics sessions were those that they were most dreading. Almost as often, these trainees are the ones who come to understand and even begin to enjoy mathematics. Is this enjoyment the point of it all? More recently, with the government requirement that trainees must be audited on their own subject knowledge and understanding of mathematics, and that they must make good any gaps that appear in a stringent list of post primary level mathematical knowledge and understanding, the anxiety sets in again. Trainee teachers realise that they must have a secure knowledge of topics which they will most probably not be required to teach. But why should this involve revisiting GSCE topics? Again these trainees often fail to understand. The answer implied in the government document is that these topics 'underpin' the material in the pupil's curriculum. This seems to silence them for a while, but sooner or later the questions arise again. Why all this mathematics?

To the best of my knowledge no one has undertaken an enquiry into the rationale of mathematics education in the detailed way in which I have done. Those writing on mathematics education do often include a short section on the reasons for teaching or learning the subject, but the reasons are usually taken for granted or glossed over. I shall be referring to several of these in my account.[1]

Before attempting to seek a rationale for mathematics education, it is worth asking why it is worth seeking. I have already made it clear that none of the answers that I have so far found are sufficiently detailed and critical. There are at least two reasons why the question 'why learn mathematics?' has become my central research question. On the one hand, in Great Britain at least, it has been argued that the aims of the National Curriculum have not been sufficiently refined, and so an examination of the particular contribution which mathematics can bring would seem to throw light on this issue. On the other hand, certain theorists have made links between a pupil's understanding of mathematics and an awareness of the purpose of the subject, which would imply that a clear idea of the value of mathematics affects pupils' ability to learn. I shall discuss each of these in turn.

The neglect of educational aims in recent years

In the first instance, my enquiry may be seen to arise as a particular case of the examination of the *general* aims of education. For one writer, at least, these have been in a state of neglect in recent years. So if this has been true of the general aims of education, perhaps it is also true of the specific area of mathematics education, in which case there is a need to examine the aims of the latter.

---

[1] In the current century the following are among those who have explicitly asked the question 'why teach / learn /do mathematics?': Fitch (1902), Yeldham (1936), Skemp (1971), Griffiths & Howson (1974), HMI (1979), Cockcroft et al (1982), Cornelius (1984), Leibeck (1984), HMI (1987), Wells (1989), Niss (1994) and Davis (1995).

At the time when John White wrote his book *The Aims of Education Restated* he wrote in the preface: 'For the past twenty years at least, the key word in writings about the content of education has been 'curriculum', not 'aims'' (White, 1982). He made this observation before the introduction of the National Curriculum in Britain. But in a later book (White, 1990) his point was reiterated. He noted that whilst a few 'bland phrases' pertaining to aims are included in the documentation, what these denote are perfectly compatible with radically differing kinds of society, and so once again, questions of suitable aims were left unaddressed. The bland phrases, in question, included the promotion of the '...spiritual, moral, cultural, mental and physical development of pupils at the school and of society', and the preparation of pupils '...for the opportunities, responsibilities and experiences of adult life' (p.14). But as White points out, such phrases might mean '...virtually anything – or nothing', and that more importantly: 'There is no obvious reason...why a tyrant like Hitler or Stalin should object to [these][2] as statements of *their* educational aims' (p.14). So White could conclude that the general aims of education, having been restated, still remain insufficiently refined by curriculum planners.

The blandness of the aims set out in the National Curriculum is in a sense understandable if not defensible. Whatever the aims of the National Curriculum ought to be, one thing is clear: they must not be those which embody or promote one particular lifestyle rather than another. Otherwise, they would seem to conflict with democratic ideals. But on the other hand, what White calls 'neutrality' is also unsatisfactory, within statements of aims, since: '...in refusing preference to some ideals of life over others, it overlooks the fact that it is tacitly presupposing the value of personal autonomy' [3] (p.22).

However, White was not daunted by either of these prima facie objections, to setting educational aims of substance, on the one hand, or omitting to state any, on the other. He argues that the demands implicit in a democratic society, which the National Curriculum must serve, do entail certain substantive aims and in his book he proceeds

---

[2] My use of square brackets is twofold. I shall use them (i) to indicate the *insertion* of my own word or words where that or those of the author are missing or require augmentation or clarification. (ii) I shall occasionally use them to bracket off words which I am retaining in the authors' original lines, but which may be ignored for my purposes.

[3] Whether of not the extended definition of numeracy, which builds in not only *ability* but also *inclination*, on behalf of the pupil, is intrusive in this respect is a nice point (see Reynolds, D et al, 1998 p.11)

to explain what these should be. So one question for *mathematics* education is whether or not it has suitable aims. Or whether it is the case that its particular aims are in a state of neglect? [4]

But even if White is right and suitable aims are scarcely to be found in the National Curriculum, the concern that the aims of recent mathematics education in Britain have been in 'a state of neglect', in the years leading up to the introduction of the National Curriculum, may seem unfounded by glancing at section 1 of the DES booklet *Mathematics from 5 to 16* (HMI, 1987). This section is clearly entitled 'the aims of mathematics teaching'. The section contains a list of aims, which is set out for all pupils in schools. It was not supposed to be an exhaustive list since it allowed for others to be added by teachers. Aims that were not left to chance were discussed under the following headings:

(1.2) Mathematics as an essential element of communication

(1.3) Mathematics as a powerful tool

(1.4) Appreciation of relationships within mathematics

(1.5) Awareness of the fascination of mathematics

(1.6) Imagination, initiative and flexibility of mind in mathematics

(1.7) Working in a systematic way

(1.8) Working independently

(1.9) Working cooperatively

(1.10) In-depth study in mathematics

(1.11) Pupils' confidence in their mathematical abilities

These phrases do not spell out a particularly satisfactory rationale for mathematics since they do not always make it explicit *why* mathematics ought to be taught, what good it is

---

[4] It is an interesting question whether the aims of mathematics could ever be other than in a state of neglect if the general aims were so. But this is not a question that I will go into.

supposed to lead to.[5] Some of the statements concern what we should be trying to do *within* mathematics. It may of course be the case that the purposes of mathematics do include the part it plays in promoting such general dispositions like self-sufficiency as in aim (1.8). But any such aim of self-sufficiency is soon neutralised by aim (1.9), that of working cooperatively, unless the overall aim is to produce pupils who can work both on their own and with others. It must be remarked, however, that such phrases are similar to the kind that White has already noted as being of a bland, unhelpful kind. Moreover, the aims headed by numbers (1.7) to (1.11) are usually regarded as desirable across the curriculum if 'mathematics' and 'mathematical' are replaced by other subject names. However, those aims that remain seem to be specific to mathematics. It is these that I shall consider first.

If we look more closely at the content of the first aim (1.2) we can see how it is determined. It starts as follows: 'Mathematics can be used to describe, to illustrate, to interpret, to predict and to explain. Above all it is used to convey meaning.' From this descriptive remark about the function of mathematics the document derives a normative claim:

> If they [pupils] can perform successfully a multiplication involving two numbers but are unable to say, when challenged, when that operation might be used, or to say whether the answer is a reasonable one or not, then there is something seriously wrong. (p.2)

Further support, for what is only an implicit aim, is given by the appeal to the 'main reason for teaching mathematics' which is given as 'its importance in the analysis and communication of information and ideas'. A similar approach is adopted in the booklet to support the next aim which is '...to encourage the effective use of mathematics as a tool in a wide range of activities within both school and adult life'. This is made to follow from the assertion that 'mathematics is a tool' by which it means that it is something which 'enables things to be done which it might otherwise be impossible or difficult to do, or to do so well'. Notice that a more *reflective* aim could have been derived from the fact that mathematics is a tool, namely, that pupils come to see that the subject has had, and can have, a particular kind of role in the advancement of civilisation. This is likely to have a completely different kind of practical implication for the teacher from that which might follow from the aim given here.

---

[5] This raises the question of whether or not there is a helpful distinction between 'aim' and 'purpose'. I

To encourage 'effective use of mathematics' would presumably involve pupils in the carrying out of certain tasks, the successful completion of which would involve the application of mathematics. The more reflective aim, on the other hand, might require a more interdisciplinary approach to the teaching of the subject, and indeed, might not even necessitate the pupils *themselves* applying any mathematics. This is not to say that such a reflective aim would necessarily conflict with the one given here, indeed the latter aim might be a sub aim of the former, but these observations do make it clear that the derivation of a normative aim from descriptive premises is not without its difficulties.

Two other aims - that pupils should have 'appreciation of relationships within mathematics' and that they should have an 'awareness of the fascination of mathematics'- are allowed to follow directly from the nature of mathematics once more. This time the fundamental point is that 'Mathematics is not an arbitrary collection of disconnected items, but has a coherent structure in which the various parts are inter-related. In very simple terms, *mathematics is about relationships*' (p.3).

The last of the booklet's aims, which is specific to mathematics, is unhelpful. Aim (1.6) on the list - 'Imagination, initiative and flexibility of mind' *in* mathematics [6] - can hardly be an aim *of* mathematics, unless, of course, an appreciation of the qualities of mind which have enabled mathematics to develop is what is intended here. If so, then we are back with something rather like the reflective aim once more. But this seems unlikely.

So far, the aims of mathematical education are derived largely from the nature and function of mathematics. Even if such a procedure is not ruled out from the start as committing the *naturalistic fallacy*, deriving aims in this way does have the usual difficulties associated with this fallacy. In particular, there is the difficulty that from any descriptive premises a *unique* practical conclusion is not entailed. The upshot of this is that whatever the nature and function of mathematics might be, in principle the way is open for deriving aims other than those set out in the booklet.

If a satisfactory account of the nature and function of mathematics cannot uniquely pin down the aims of the subject for educational purposes, perhaps the reasons for teaching

---

will discuss this presently

the subject will do this, particularly if they are of a normative kind.[7] (Recall that the first aim from the above list was linked to 'the main reason for teaching mathematics'). Yet even when the question 'why teach mathematics?' has been asked it does not always make it clear *how* mathematics can attain the ends given. For example, in a section titled 'Why teach mathematics?' in a chapter from a book on the teaching of mathematics in a secondary school, Cornelius (1982), suggests that the reasons for teaching mathematics are simply a matter of consensus amongst those who teach it. He writes:

> Only if some broad measure of agreement can be reached in answer to the question 'Why teach mathematics?' can we begin to discuss *what* we teach and *how* we might plan and organise the teaching (p.38).

If by this remark Cornelius is claiming that such agreement is a *necessary* condition for making decisions about what to teach, then this is surely true. But whether it is *sufficient* is another matter.

Cornelius gives the following list of aims derived from a group of secondary teachers:

> (1) to develop the ability to think, communicate and reason clearly and logically,
>
> (2) to provide tools and skills necessary for use in the real world, everyday life and other subjects,
>
> (3) to develop the ability to recognise patterns and relationships and to generalise from experience (including the use of symbols)
>
> (4) to develop creative ability,
>
> (5) to increase awareness of other cultures and interest in the world. (p.38)

He asserts that these general aims would probably receive 'fairly wide acceptance' among secondary school teachers, and the problem he goes on to discuss is the inevitable gap between aims and what happens in practice. But suppose that this list does constitute a rationale, which has a 'broad measure of agreement', or 'widespread acceptance' *among secondary school teachers,* will this settle the matter? Can we then simply turn our attention onto considerations of methods and content? Surely few

---

[6] My italics

secondary teachers would disagree with such reasons. But why should it be supposed that the question of aims rests simply on a matter of agreement between teachers of mathematics? One difficulty with relying upon such agreement among mathematics teachers in setting the aims of mathematics education, is that the agreement could be reached by setting aims which were so bland that few would want to disagree with them. The above list hardly seems to be unacceptable to most people. Surely only a Hitler, a Stalin (or a Milosevic) would have been less than happy to acknowledge that one reason for teaching mathematics should be 'to increase awareness of other cultures and interest in the world'.

Another difficulty is that even though some reasons given for learning mathematics may not conflict with general educational ideals, it may still be highly questionable whether mathematics is *fit* to realise the ends embodied in such reasons. 'Creative ability',[8] for example, mentioned in the penultimate item of Cornelius' list might well be agreed by all to be a laudable *educational* aim. But it is certainly not obvious that such ability really can be developed through the study of *mathematics* or that it is effectively developed in this way. If creative ability can be developed through the study of mathematics then it at least needs to be explained how?

Comparing the way that the HMI Curriculum 5-13 booklet determines its aims with the way Cornelius justifies the teaching of mathematics we seem to have two contrasting approaches. To put the matter starkly: *Either we value certain aspects which we believe are in the nature of mathematics which then become translated into reasons for teaching the subject,* **or** *we value certain things independently of mathematics and assume that in learning mathematics these will be attained.* Few writers in recent years seem to strike a balance between these two extremes and it is perhaps in this respect that the aims of mathematics education are, as White has said of the general aims, in a state of neglect. Griffiths and Howson (1974) are exceptional in beginning to make connections between the two approaches. After surveying a variety of reasons for

---

[7] I realise that there is the possibility of course that any normative reason for teaching mathematics might already be so closely attached to a particular aim to render the search for an aim futile.

[8] Even 'creative ability' does not fly in the face of the educational ideals of some regimes, my point remains. Even if the most austere of regimes welcomes a particular good which it is supposed that mathematics can bring, acceptance of the good in no way guarantees that mathematics is fit for the purpose. I shall explore *some* of the issues surrounding creativity and mathematics in Part 2.

learning mathematics they make explicit reference to the idea that mathematics can train the mind in some way when they write:

> ... it is worth drawing attention to the essential difference between the 'training ground' aim and the others. It alone is not aimed at producing behaviour or responses that can be labelled specifically 'mathematical'. Rather is mathematics being used as one of the most suitable means for the inculcation of ideas and ways of thought considered educationally desirable.
>
> Far too often when considering why we should teach mathematics, we forget that education has wider aims than are immediately apparent from the school timetable. For example, the supporters of the method of teaching children in small groups rather than as a class frequently stress the subsequent improvement in mathematical learning, but rather less often do they argue that it is teaching children to work cooperatively in small groups which is probably the more important educational aim. (p.23)

Whether or nor mathematics fulfils this training role is not shown adequately by the authors and one of the major tasks of my enquiry is to see whether mathematics is fit for such a purpose. Accordingly, I do hope to maintain a balance by focusing upon certain educational ends, on the one hand, and trying to see whether the nature of mathematics is fit to achieve the goods implicit in such ends, on the other. I shall be considering two purposes of mathematics in detail, but as we shall see these are extensive. They centre around art and aesthetics, in Part 2, and mental training in Part 3. But in order to distinguish these purposes from the perennial belief in the utility of mathematics, this and its many varieties will be outlined in Part 1. However, before moving on to the other reason for my enquiry, some further clarification is necessary.

Aims and Purposes

So far I have been discussing aims in mathematics education and general education as though these were equivalent to purposes. However, it seems that there is an important distinction between the two concepts of aims and purposes, which Moore (1982) believes:

> ...may best be brought out by drawing attention to two different questions which may be put to someone who is engaged in a practical task. The questions are: what are you doing? and: what are you doing it for? To take the second of these questions first, to ask: what are you doing it for? is to presuppose some end outside the activity itself, which the activity is designed and intended to bring about...the answer is given in instrumental terms... A rather different approach is indicated in the first question: What are you doing? Here someone is being asked to specify what his action is, to state its content. (p.27)

This distinction which Moore has drawn out is indeed one which has been employed in the context of mathematics education. In the document *Mathematics 5 to 11: A Handbook of Suggestions* (HMI, 1979), for example, an explicit distinction is made between three aspects: 'purposes of teaching mathematics', 'mathematical aims', and 'objectives'. The authors of the document do appear to be using 'purposes' to refer to goods which are in a sense outside mathematics, when they write:

> The need for schools, both primary and secondary, to provide their pupils with a good foundation of mathematics has been argued on three main grounds. Mathematics is useful, mathematics is part of our culture, and mathematics trains the mind. There are many other arguments but they nearly always reduce to variations of these three; and the three interrelate with one another. (p.4)

In accordance with what Moore has said, above, the HMI document does regard *aims* as reflecting what it is we ought to be trying to do *within* mathematics. That is, they are 'essentially declarations of intent that give direction and shape to a scheme of work or teaching programme'(HMI, 1979 p5).

Finally, 'objectives', in the same document are meant to refer to:

> ...detailed descriptions of performance which teachers can accept as evidence of learning and as 'milestones' along the path of progression and development. (p.5)

Where pressing questions in mathematical education are being asked, too, Moore's distinction is implicit. For example Costello (1993) writes:

> There are plenty of people who believe that mathematics education has lost its way. Newspaper reports in recent months confirm this feeling; but the saddest part of this concern is the suggestion that we need to go back – to recover some perceived strengths of a previous age. Of course, this is irrational prejudice. But it does oblige those of us concerned with teaching mathematics to examine our intentions. We need to ask "*What is progress?*", or even more pretentiously "*What is it that mathematics education has to contribute to civilisation?*"

In asking for the contribution which mathematics education can make to civilisation Costello is seeking something *outside* mathematics and hence is making a request for its *purpose.* But to 'examine our intentions' would suggest that clarification of what we are trying to do *within* mathematics education is what is sought and hence he is seeking the aims of the subject.

The area which I am concerned with, falls within the rationale for mathematical education, its point, aims, purposes, goals or justification. Notwithstanding Moore's distinction, I shall treat all of these as objects of enquiry for the moment. I shall also treat as objects of enquiry some of the questions which are used to *elicit* aims, purposes, goals, such as 'why is mathematics necessary?' 'Why learn mathematics?' 'Why teach mathematics?' 'Why do we need mathematics?' and so on. Only when these types of questions fail to elicit helpful answers that I am seeking will I make a distinction between them.

Understanding and Purpose

There is, as I have already mentioned, another reason why the aims and purposes for learning mathematics need to be clearly spelt out, and this directly concerns the individual who is trying to learn the subject. If mathematics is often a difficult subject to learn, as I have insisted it is, then one way to make it easier to learn would be to know more about how it is best learnt. Whilst there may be a range of answers to this question involving, say, the most expedient teaching resources, and the most effective structuring of the content, certain writers have recently drawn our attention to an important connection between understanding a subject and beliefs about its purpose. These writers suggest that understanding mathematics and knowing the purpose of learning it are *internally* linked. In saying this, such writers are not simply saying that there is some expedience in ensuring that someone sees the point of learning mathematics, that it provides some kind of motivation for learning the subject. What is being suggested is that understanding mathematics is partly a function of being aware of what it is *for*.

Munn (1997), Sierpinska (1994) and Passmore (1980) all seem to me to be making essentially this same point: understanding and purpose are internally linked in the learning of a subject. Munn argues, from an empirical standpoint, that the children's ability to count is a partly a function of their understanding the purpose of counting as a means of quantification. Sierpinska arguing more conceptually sets her sights more on topics in higher mathematics where she wants to separate off an objective criterion of understanding from a subjective idea of seeing the point of a particular theory. Finally, in a more general way, Passmore connects the case of a pupil's 'not seeing the point' of a subject as one way in which such a pupil does not understand it, thereby making a conceptual point between cognition and purpose. I shall discuss each of these in turn.

Munn (1997), setting out to explain a contradiction within the specific area of learning to count, writes:

> Currently, researchers are alleging that young children are competent with number, yet simple observation shows that children miss the most obvious implications of number logic until they are older. (p.10)

She gives three examples of these 'implications' which are missed by young children. These are their inability to count objects when they are randomly arranged, their failure to understand the conservation of number, and their tendency to judge the relative quantities by sight. However, to rely on this evidence, as she complains that researchers do, is to focus on what Munn calls 'external' factors. She believes that the apparent contradiction can be removed by casting our attention towards the *subjective* considerations of pupils' purposes for counting. She argues that what have been seen as errors in *understanding,* are internally linked with the *purposes* of counting. Munn, in her research, asks the following questions to the children: 'Can you count?' 'Can you count these blocks?' 'Could you give me (one, two, three, etc.) blocks?' 'Do you count at home?' / 'What do you count at home?' She does this in order to gain some insight into children's *beliefs* about counting. She remarks that:

> The children's responses to the question 'What do you count?' and 'Why do you count?' showed that it was very rare for them to understand the adult purpose of counting before they went to school. After school entry, some children seemed to have gained a little understanding, but it was uncommon to have a child respond to the 'Why?' question by saying 'To know how many'. This was true even for those children who were quite competent at the counting involved in giving nine or ten blocks. They could count in all three senses: saying the words, linking the words with the objects, and linking the last word in the sequence with the amount. Yet only one of the children at the pre-school visits responded to the 'Why' question by replying 'To know how many'. The rest of the children showed by their responses that they were perplexed by the notion of counting as a purposeful activity. (p.13)

The pupils did have their *own* reasons for counting, such as enjoyment – what Munn calls 'counting to please the self' – others were counting simply as part of a social activity. The upshot of this, is that development in children's counting is a partly a function of their awareness of its purpose. Remarking on the striking difference between children's understanding before they start school and afterwards, she adds:

> The progression outlined here demonstrates the influence that social beliefs can have on cognitive development. There was a clear disjunction between the children's counting behaviour and their beliefs about counting.[9] (p.16)

Munn suggests in her implications for teaching, that teachers should 'make the purposes of counting explicit for children' and also that they should 'stimulate children to develop their own numerical goals'. If she is right in her conclusions about the elementary aspect of counting, then there are good reasons to suppose that throughout mathematics pupils' beliefs in the purposes of mathematics go hand in hand with their understanding of the subject.

Sierpinska (1994) seems to be making a similar point. Whether or not, like Munn, she is making an empirical point, or whether she is trying to establish a conceptual connection between understanding and purposes is not altogether clear. She certainly discusses a transcript of a conversation in support of her argument in which she makes a distinction between the notion of understanding per se and the normative notion of *good* understanding. Some of her assertions, however, certainly seem to be independent of empirical confirmation. Like Munn she argues that understanding is often judged by external factors when, for example, it is a matter of, determining whether what a student knows is consistent with a particular theory which he or she is learning. But she writes:

---

[9] My italics

> When it comes to understanding not a particular concept of a theory or a particular method but the theory as a whole, when, for example, one asks the question 'what is the point of this theory?' then the evaluation must be more subjective. Here the problem is not so much with the meanings as with the significance, and the criteria of significance are not a matter of just the logic. The judgement depends upon one's philosophical attitudes towards scientific knowledge, views on the *raison d'être* of the theorizing thought, on the goals of learning mathematics, on one's theory of intellectual development, etc. The judgement of a person's way of understanding will be relative to cultural norms, which are not justified by reference to some logical system but by an appeal to traditional values. (p.113)

In contrast to Munn, Sierpinska locates her view in advanced mathematics where the understanding of a particular *theory* is at issue. Nevertheless, like Munn she is insisting that the pupil must somehow come to appreciate the value of a particular theory *within* mathematics in order to achieve understanding, just as for Munn the child has to appreciate that the particular activity of counting *within* mathematics has quantification as its fundamental purpose. I have emphasised the fact that what both writers are drawing out are aspects *of* mathematics which are linked to purposes. This is an important point. It is one thing to admit that there are some aspects within the subject that may seem pointless, another to question the value of the subject as a whole. However, Sierpinska's mention of 'the goals of learning mathematics' does suggest that the pupil's problem is part of the more general question: 'what is the point of mathematics?'

There seems to be, then, at least grounds for supposing that an empirical connection exists between *cognition and purpose* within some aspects of mathematics. There is also the suggestion that there may even be a general conceptual connection. However, the latter possibility is brought out more strongly in another writer, John Passmore, for whom being clear about the purpose of mathematics is part of what constitutes *understanding* of the subject. Passmore (1980), states four different ways in which a pupil may not understand something, namely:

> ...when he misunderstands, fails to understand, half-understands, or *sees no need to understand.* [10] (p.198)

It is the last of these four ways which is important here. It must be admitted, however, that the notion of 'seeing no need to understand' is at first associated, by Passmore, with

---

[10] My italics

the pupil who takes things for granted. So that the corrective action for the teacher becomes a matter of *puzzling* the pupil in order to shake him or her out of a complacent state. Yet further on in his account, Passmore interprets this lack of understanding as explicitly involving the *point* of learning a particular subject. In which case some kind of justification is required. He writes:

> We no longer ask 'What is the point of rainbows?' or 'What is the point of the earth going around the sun?' But in other cases, it is still a perfectly proper question ...It is certainly proper in relation to the whole apparatus of schooling, the studying of particular subjects, examinations and so on. (pp. 203-4)

Whether puzzling the pupil best brings this about is another matter. In Plato's *Meno* (1956 edn) we could describe the slave as being in a complacent state about his knowledge, before Socrates gradually puzzles him and brings him to realise that he did not know what he believed he knew. There would of course, then, be some reason for teaching him what he did not know. But, ignorance cannot be a sufficient reason for learning, otherwise the demand for a justification for learning anything would be unnecessary. What the pupil who asks why s/he should study mathematics wants to be shown is, as Passmore puts it: 'Either that mathematics has a value in itself or that it is a means to some end he accepts as being worthwhile'. This dichotomy - that something either has value in itself, or because it is a means to some valuable end - can be traced back at least as far as Aristotle's *Ethics*. It is implicit in the contrast of purposes drawn out by Munn, above, where children carry out counting for pleasure perhaps oblivious to the fact that for others it is simply a means to quantification. It will reappear in the discussion below. But for Passmore there is a conceptual connection at stake. He adds:

> 'I don't understand the *point* of ...' is indeed a very common form of not understanding. And often enough...*the answer is far from being obvious.*[11] (p.205)

My second reason for carrying out this enquiry is thus to attempt to throw some light on the non-obvious question regarding the point of mathematics which, it appears, prevents or limits understanding of the subject on behalf of the pupil.

---

[11] My italics

The structure of the thesis

I have already remarked that most people would agree that there is some mathematics that must be learnt for everyday use. The notion of utility will therefore be my first point of departure. Since, it cannot be gainsaid that mathematics has been used for a number of purposes in shaping the world in which we live, I shall not be attempting to argue that mathematics is scarcely useful. What I shall be more concerned with is asking whether there are any other grounds for learning mathematics apart from its usefulness. In Part 1 I shall outline what I see as the non-controversial ways in which mathematics may be said to be useful to everyone, before extending this discussion towards other ways in which mathematics is useful yet not necessarily *directly* useful for everyone. The idea of usefulness being distinguished by its immediacy or directness seems to be one way of distinguishing between *kinds* of reason. Mere practical utility, it seems, is distinguishable from other kinds of utility such as the 'use' of mathematics for improving or training the mind in some way. But whether or not mathematics can have value and purpose without any reference to the notion of usefulness, whether such usefulness is immediate, direct, practical or whatever, is something else which I shall also explore in Part 1.

There have certainly been many who have thought that the subject may be learnt for *intrinsic* reasons. But since intrinsic reasons are often described in terms of 'doing something for its own sake', this notion will be explored carefully and will reveal that the expression can either mean pursuing something primarily for the benefit of the individual or primarily for the benefit of the discipline. Only in the first sense can the 'doing something for its own sake' be free of *all* taint of usefulness. This is because if a subject is pursued for the sake of the discipline then this implies that two other senses of usefulness may be introduced namely usefulness of something within a discipline and deferred usefulness of the discipline at some indeterminate time. As the intrinsic reasons for learning mathematics are somewhat eroded by a broadening of 'usefulness' I shall illustrate how some writers have attempted to dispose of any kinds of reasons except utility and hence that there are not different kinds of reasons for learning mathematics, only different ways in which it is useful. This holistic approach, I shall argue, is unsatisfactory as it side-steps and marginalises what have been regarded as good reasons for mathematics for centuries.

In what I have called the Interlude I set the stage for two particular reasons for learning mathematics which will be tackled in depth in Parts 2 and 3. Certain reasons which are *beyond utility,* then, form Part 2, Art and Aesthetics in Mathematics Education, and Part 3 Mathematics as Mental Training. It is because both of these reasons stem ultimately from doctrines which can be traced back to Pythagoras that he is introduced and set aside in this Interlude before these latter parts begin. Following certain writers, I characterise Pythagoras as being responsible for a non-materialistic view of the universe and thus a particular view of the goodness of the human being from which the notion of 'purification' is central. The ancient notion of 'purification' was importantly achieved through a particular approach to mathematics - the contemplative approach - and therefore becomes the precursor of mathematics as the improvement of the soul. This contemplative approach has echoes in both Parts 2 and 3. In Part 2 the connection with contemplation is made by those who want to view mathematics as a fine art, and in Part 3 by those who believe that reflecting upon special properties of mathematics leads to an improvement in the mind, in short that mathematics trains the mind.

In Part 2 I try and find out what connections could possibly exist between mathematics and art and thus whether mathematics could be classified as a fine art in some way or other. After showing that the claim that mathematics is fine art cannot be sustained I do allow a more modest claim to be sustained: it may still have aesthetic aspects of various kinds and thus plausibly be a source of aesthetic experience to those who engage in it. To accomplish this it is necessary to show that art and aesthetics are indeed separable and even then to narrow down those ways in which mathematics may be viewed from an aesthetic point of view. These include the considerations of pattern in mathematics, proof and the origination of mathematics from first principles. All of these I argue can be seen to have an aesthetic dimension.

Part 3 looks at the other reason for mathematics which is beyond utility – mental training. In this part I show how both the contemplative aspects, stemming from Pythagoras, and what have often been introduced as a contrast - discursive aspects - (or what I call the procedural aspects stemming from Descartes) are both at work. Neither view is completely successful, yet perhaps surprisingly the contemplative view is either less challenged or is less resistant to being criticised. Its outcome is, however, difficult to clarify. In a similar way to Part 2, I introduce a more modest view of mental training which in one form or other has had a long history. However, whilst the view is clear and crisp in its outcome it is nevertheless hard to show why mathematics is ultimately the best means to it.

Part 1 - The Bounds of Utility in Mathematics Education

Chapter 1 – How Mathematics is useful

Quantities and indirectness

Let us begin with the classic dichotomy of value, introduced earlier by John Passmore, that something might either have a value 'in itself' or as a 'means to some end' which is supposed to be of value. I said that we can trace this notion back to antiquity, and this is a helpful place to take up the discussion at this point.[12] From ancient times a fundamental desire for individuals and societies to survive and flourish, has provided a motivation for the development of mathematics. But precisely what counts as 'flourishing' is not easy to determine. Two broad views of the good life are however identifiable. For many philosophers it has centred round a life in pursuit of wisdom characterised by *contemplation*. This may be contrasted with the life of *action*, which included not only the life of ordinary work but also a life spent in pursuit of fame and glory in conquest and battle. Both of these ways in which human flourishing was believed to consist, relate to the pursuit of mathematics in different ways, and have provided reasons for learning it.

Of course not everyone, then as now, was able to approach mathematics in both these ways. The demands of the situation in which one finds oneself, may prevent one from having the necessary resources to sustain a life of contemplation. So that only those, for example, who were secure, self-sufficient and had leisure time, placed intrinsic value on the subject. For others the subject was rather more like a tool which assisted the life of action. Flourishing in this latter respect puts a premium on finding quick and efficient ways of getting things done. Mathematics was found to be just such a tool for this purpose and was therefore actively sought, developed and passed on. According to Kline (1980), the origins of mathematics to meet *practical* demands of this kind were particularly associated with the early Egyptian and the Babylonian civilisations. Yet at this stage in its development Kline informs us that:

---

[12] Passmore gives his own view of the utility of certain areas of mathematics when he discusses what he calls 'open' and 'closed', 'broad' and 'narrow' *capacities*. (See Passmore, 1980)

> ...mathematics was hardly a distinct discipline – it had no methodology nor was it pursued for other than immediate, practical ends. It was a tool, a series of disconnected, simple rules which enabled people to answer questions of daily life: calendar-reckoning, agriculture, and commerce. These rules were arrived at by trial and error, experience, and simple observation, and many were only approximately correct. (p.18)

But as mathematics did become a distinct discipline, its practical value has remained of paramount importance in education. Despite the great differences between the fundamental philosophical positions of Plato in antiquity and Locke of the Enlightenment, they both agree on the educational value of mathematics. Thus the Athenian remarks in Plato (1970 edn):

> A man...will fall a long way short of [such] godlike standards if he can't recognize one, two and three, or odd and even numbers in general, or hasn't the faintest notion how to count, or can't reckon up the days and nights, and is ignorant of the revolutions of the sun and moon and other heavenly bodies. It's downright stupid to expect that anyone who wants to make the slightest progress in the highest branches of knowledge can afford to ignore any of these subjects. (p.312)

Locke (1902 edn) writes in a similar way:

> *Arithmetick* is the easiest, and consequently the first Sort of abstract Reasoning, which the Mind commonly bears, or accustoms itself to: And is of so general Use in all Parts of Life and Business, that scarce any Thing is to be done without it. This is certain, a Man cannot have too much of it, nor too perfectly: He should therefore begin to be exercis'd in *Counting*, as soon, and as far, as he is capable of it; and do something in it every Day, till he is Master of the Art of *Numbers*. (p.157)

Locke's plea at the end of the seventeenth century is as convincing now as it was then. In Britain, at least, the recent National Numeracy Strategy has recommended a daily mathematics session throughout primary schools. Yet something that is interesting in the remarks of both Plato and Locke is the suggestion that it is *counting* which is fundamental. In recent times the view that number is a property of sets, has given way to the idea that a set is at least *logically* more fundamental than the idea of numbers or counting. Nevertheless, despite the influence that this has had on the curriculum there is now some evidence that the emphasis should be upon counting. So that counting provides once again the bedrock upon which the usefulness of mathematics is based. This much we have seen is already implicit in Munn's discussion of purposes in very young children (see page 16). It is worth adding perhaps, that even though we might be prepared to insist that mathematics *begins* with counting, most educationists are concerned that pupils do not rely solely on counting methods. So counting can no more be *the* reason for learning mathematics any more than the learning of scales could be *the* reason for learning to play a musical instrument.[13] But it does seem to be the clearest way of beginning an argument to show that mathematics is *useful*.

If, as Quine (1969) has remarked, 'we are prone to talk and think of objects' (p.1) rather than persisting in seeing the world as only stuff, then the need to keep tabs on such objects seems to be inescapable. We can keep tabs on objects in ways other than counting simply by comparing sets of objects with other sets of objects. Yet counting makes the task so much easier. Counting, then, is a convenient *tool* for finding quantities, and this ability to determine quantities provides *one* of the first uncontroversial reasons for learning mathematics. Even quantities of continuous stuff may be established by the counting of units, although the use of the *real* numbers, rather than simply whole numbers, is a further useful refinement in measurement. Counting is one of the most elementary and general ways in which this can be done, and a remarkable one too, which took many years to develop.

At an even more elementary level than counting it is possible to see the practical value of mathematics. Suppose that we wish to acquire some chairs for a meeting. We could ensure that we have enough by arranging matters so that every member was asked to help him or herself to a chair from a stack, so that each person would be paired with one chair. The quantity of chairs required is then *equivalent* to the number of people. 'Equivalence' here means that there is a one-to-one correspondence of people to chairs. This method is perfectly satisfactory for determining quantities. But it is important to note that it uses a method of *direct comparison*. It requires physically juxtaposing the set of people and the set of chairs.

But the idea of equivalence is the precursor of counting. Counting allows us to make a correspondence between one collection and an ordered set of number words. The last number in the count provides us with a name, the cardinal number, of the collection in

---

[13] This comparison is not altogether satisfactory. The value of counting lies in part at least *outside* mathematics, this is not true of scales no application of which seems to be made outside music. Nevertheless, scales do seem to have the same kind of fundamental role in music as they do in mathematics.

question, provided that one word is paired with precisely one of the objects of the collection, the quantity of which we are trying to determine. The connection between counting and the more primitive method of comparing objects can be shown if we recover the set of words used for any count. Thus, if I count my fingers and give the number 'five' as the cardinal number, I can easily recover an equivalent set for this comparison. It is the set of words {one, two, three, four, five}. The size of this set of words is equivalent to the set of my fingers.

So counting, at the very least, allows us to carry out comparisons *indirectly*. Continuing with the earlier example: we can count the number of people, and without recovering the set of names used for this process, note the *cardinal* number, the last number used. Then the chairs can be counted out until the same cardinal number is reached. The ordered set of number words are used as a kind of 'go-between', but for the cardinal number itself to have any meaning and use, rather than as an equivalent set of number names which can always in principle be recovered from any count, some sustained learning is required.

Thus we have reasons for learning the number names, the practice of pairing and the appreciation that the last name used gives the *quantity* of the collection involved. One must acquire, too, a sense of the size of numbers; to know, for example, that 30 is greater than 3 but less than 300, and also that 9999 is considerably larger than 99. In this way educators have good reasons for teaching counting and the relative size of numbers, at the very least. Children, too, should be able to realise that mathematics can be used to obtain quantities. Moreover, we can say that counting, because it can save us the trouble of making certain *physical* comparisons between collections, can be regarded as a *tool* for quantification. On the other hand, if the notion of a tool seems to have physical connotations, we may prefer to emphasise the symbolic aspect of counting which then may be regarded as constituting a kind of language in which quantities are communicated.

If counting provides one example of how mathematics can be useful by allowing us to quantify *indirectly*, that is, with the minimum amount of manipulating of physical objects, there are many other examples. Much more than this can be achieved by considering the application of the operations of simple addition, subtraction, multiplication and division. With these we can answer hypothetical questions like 'how many chairs would be required if, each member brought a partner, say, to the meeting.

Or perhaps 'how many chairs would be required if twelve people were unable to attend?' In the case of such hypothetical cases we find ourselves relying more on the manipulation of *numbers* than objects even though we could still continue to set up physical representations of some kind. It is at this point that questions can arise about the nature of numbers. Since they are no longer objects of perception their exact nature became a subject of study and hence we have the possibility of focusing our attention on the abstract relationships between numbers and their properties. This change of focus, as we shall see, provides a reason for pursuing mathematics independently of its usefulness.

A similar account to that of counting can be given of *measuring*. This can in many cases be carried out by direct comparison, but the development of the real numbers allows indirect comparison to be carried out. The notion of directness and indirectness runs deep in mathematics and so the metaphor of 'tool' is appropriately applied as a description of mathematics when, for example, quantities of collections are determined with the minimum amount of physical contact. Counting and measuring are simple examples of this idea, but a further example is in the use of tables. Mathematical tables can be set up to represent extra-mathematical objects by another central idea in mathematics, the idea of an isomorphism. Finally, whole mathematical models can be constructed to represent certain aspects of extra-mathematical situations.

Utility in wider contexts

Beyond the quantification of the objects in routine everyday life, mathematics has had practical value when states of emergency have been anticipated. In *The Republic* Plato (1955 edn) emphasises the importance of numbers for the security of the state when he remarks that 'soldiers must study them so that they can organise their armies...' (p.332). In eighteenth century England, too, where, interestingly, the practical value of mathematics in education had often been neglected, one of the key landmarks was the founding of The Royal Military Academy at Woolwich. Howson documents an important event leading to the realisation of the need for such education:

> In 1662 a request was made for a schoolmaster to be supplied to the troops quartered in Madras. The schoolmaster was to divide his attention between the soldiers and their families. Gradually the idea of supplying education to the troops spread and in the mid eighteenth century the first army schoolmaster was appointed in England. (Howson, 1982 pp. 64-65)

However, by this time rather more mathematics than the determining of quantities was expected of some individuals:

> For officers... and, in particular, staff officers... specialised knowledge of mathematics, fortification and drawing was necessary. To meet that need the Royal Military Academy (RMA) was established at Woolwich in1741 to train future artillery and engineer officers. (Howson, 1982 p.65)

Other changes, too, became dependent upon mathematics education. Howson documents the gradual importance that mathematics was to have in navigation for the great sea voyages. This was to lead up to another landmark in English mathematical education: the assistance of Samuel Pepys in founding the Christ's Hospital School. Pepys worked at the Admiralty, where he found himself lacking in arithmetic. Howson recounts another important set of events in English mathematics education history that arose from this:

> ... Pepys gradually came to realise that not only he, but the Navy as a whole, was handicapped by a lack of mathematical expertise. The restoration of the monarchy in 1660 had been quickly followed by a naval war with Holland which seriously depleted the complement of naval officers. It was necessary to make good these losses and it was increasingly clear that among the replacements should be many officers who had a good grasp of navigation and the underlying mathematics. (Howson, 1982 p.35)

But if mathematics has an important use where national security is concerned it has also been of value in commerce. Yeldham (1936) describes the effects not so much of counting but of the new rules of reckoning or algorisms developed in Hindu-Arabic arithmetic. These methods of calculation were acquired and used by the merchants. Most of the earliest mathematics textbooks consisted largely of discussions on the properties of numbers - the *theory* of arithmetic - rather than the art of calculating. Gradually applications began to appear in newer textbooks particularly by way of worked examples and exercises from commerce. So that as Yeldham puts it: 'By 1600, sheer usefulness had won the day', implying that mathematics had not always been pursued for practical ends.[14] But in being selective in the kinds of contexts used in textbook exercises there is always the danger that mathematics will appear to favour

---

[14] It is important, for the discussion which follows, to notice how writers make this implicit common sense distinction between 'usefulness' and what for the moment can only be denoted as the 'non usefulness'

particular groups in society. This seems to have happened as mathematics texts were filled with examples set in commercial contexts. Thus Yeldham remarks:

> [This] desire to be of help to men in trade went too far, arithmetic becoming so commercialized that in the seventeenth century it was being neglected except by those whose way of life demanded it. (Yeldham, 1936 p.13)

When the utility of mathematics is pushed too far in a particular direction, like this, the way is open for emphasising a different aspect of the subject. But before pursuing any divergence in the utility rationale for mathematical study, we should consider one other important and fairly uncontroversial sense in which mathematics is deemed to be useful.

The use of mathematics 'across the curriculum'

It has already been suggested in the passage from Locke, above, (see page 24) and is undoubtedly accepted by most educationists, that mathematics is of use within other subjects of study. Mathematics is useful, say, in enabling statistical information to further the study of science. In being able to calculate the half-life of a substance mathematics is often useful within history. Perhaps, too, we find measuring techniques indispensable in learning geography, or making certain aspects of music intelligible. Again, the mathematical theory of perspective geometry, whilst originating in the art of drawing and painting three-dimensional representations, may be said to be of use in the theory of techniques in art.

Notice that the application of mathematics in each of these, and many other similar cases, is different from its use in the examples discussed earlier of everyday life, states of emergency and commerce. To say that mathematics is useful *within* other subjects suggests that the mathematics might not be *directly* useful. Mathematics may be useful in astrology. But suppose astrology is shown to be useless does this now show that mathematics is not as useful as was originally supposed? It is certainly true that within an area of mathematics, number theory for example, for which scarcely any application might be made, a particular result may nonetheless be very useful. So, even if it would be difficult to show that any of the traditional subjects which make use of mathematics are themselves of dubious practical use, the point remains that the usefulness of mathematics in other subjects may be attributed somewhat independently of their worth. Yet in general when it is claimed that mathematics is useful in other subjects it is supposed that on the whole these other subjects are worth pursuing. But even if the curriculum were of the most liberal kind, the use of mathematics across the curriculum in the most 'useless' of subjects could still be upheld.[15]

A divergence within the rationale

As mathematics became a systematic discipline it was characterised in two ways. Put succinctly, there was the theory of numbers and there was the art of calculation. This has seemed to allow a branching of the subject's aims. When the practical side has seemed to be too specific or in later times even lacking in rigour, the theoretical side becomes important and different reasons for pursuing mathematics arise. When preoccupation with the theoretical aspects of the subject leads to a lack of success in the life of action then the usefulness of mathematics is restated. If we refer back to the two quotations from the contrasting major philosophers, (see page 24) we can see that this distinction lies just below the surface, as it were. For Plato, there is not only the suggestion that one should be able to 'reckon up' or calculate, but also the suggestion that we should know certain properties of numbers. This fairly innocuous sounding distinction was one of considerable importance in the history of the subject, as the Greeks distinguished between logistic and arithmetic – roughly the art of calculation and the theory of number.[16]

Locke, too, (see page 24) draws out a further point about mathematics which he announces at once, namely, that mathematics is a form of abstract reasoning. It is not clear, in the fragment which he gives us, whether this abstract reasoning of mathematics simply provides the necessary condition for counting, or whether the reasoning itself has special value. Yet many other writers have indeed believed that the abstract reasoning involved in mathematics is of value in its own right, and that this provides something of a by-product which justifies the learning of the subject in a distinctive way. Plato, in *The Republic*, and other more recent writers, have argued in this way that there are special properties of mathematics which provide additional and pressing reasons for learning the subject. Thus we shall see that a range of aspects of mathematics are supposed to provide different *kinds* of reason for learning mathematics,

---

[15] On a related issue of this kind, Wringe has pointed out that 'for something to be good as a means it is only necessary that there should be other things which we want to do for their own sake'. (Wringe, 1988 p.123) He goes on to suggest some things which we may do for their own sake but which we need not consider valuable. I shall discuss the notion of doing something for its own sake below. All I want to point out here is how something may be a good means, i.e. useful, without its having a valuable end.

ones which ebb and flow in priority within the justifications which have been set out for the educational value of mathematics. We shall now examine in some detail how these different kinds of reasons have been distinguished by some writers whilst others have ignored or refused to acknowledge that such distinctions exist.

[16] This (rather confused) distinction is discussed in detail in Klein (1968)

Chapter 2. – The basis for different kinds *of reasons.*

We have seen that mathematics is useful in various ways. The ways in which it is useful do not of course form an exhaustive list. All I have tried to do is to indicate one or two of the main uses which are often set out. New practical applications have been found for mathematics throughout history, and one may safely assume that this will continue. Indeed much mathematics is *originated* in order be of assistance to those who are working at practical tasks. In this section I am no longer concerned to find more and more examples of the uses to which mathematics may be put. I want to explore the possibility that there are different *kinds* of reasons, other than utility, which provide us with the purposes of mathematics and hence a reason for teaching it. We shall see that much depends upon the *scope* of what counts as 'useful' and this varies amongst different writers.

Unitary reasons

When reasons are given for learning mathematics these are sometimes in the form of a list of reasons, all of which are of a single broad kind. So that if mathematics is deemed to be *useful*, as it has already been argued, the reasons given for learning it may all be examples of the use to which mathematics may be put. So that mathematics is typically said to be worth learning because it is useful, for example, in everyday life, in commerce, states of emergency and in the development of science. These four reasons would be all of the same kind. I shall call such reasons *unitary*, since they can all be traced back to one main source.

On the other hand, different *kinds* of reasons for learning mathematics may be sought and put forward. Some writers have thought that the value of mathematics is *not* to be found solely in its usefulness. Others have even denied that mathematics is of much use at all, or perhaps even if it is, it is not primarily *this* reason which provides a rationale for teaching and learning it.[17] So if it is to be justified, by those who have these beliefs about the subject, it is necessary to have some means of giving reasons which are marked off from what is useful. But the notion of what is useful is elastic, and this has allowed some interesting variations in the way writers have responded to the possibility of kinds of reasons.

[17] See for example Andrews (1998).

Broadly, mathematics is useful to the extent that it is the *means* to some desired end. Typically, as we have seen, these ends include those connected with national security when the value of mathematics in military goals has been realised, and those connected with prosperity when mathematics has seen to be of value in commerce. Overlapping both of these, the use of mathematics in navigation has provided strong grounds for teaching and learning it. Additionally, mathematics is seen as being of indispensable service to science. The most clear-cut cases of usefulness, then, involve ends that lie outside mathematics. However, since some results, techniques, arguments and proofs are useful means for achieving or justifying other results, certain parts of mathematics become useful *within* the discipline itself. A theorem might be useful in showing the certainty of some mathematical proposition which hitherto was only conjecture, and moreover it may be useful for bringing together a range of other results or generating further ones.

Another idea which some have thought to be distinct from all of those mentioned so far, is the function that mathematics has in achieving certain ends outside mathematics, but 'within' the individual. This end is described in various ways, but I shall denote it by the expression *mental training*. A full discussion of this reason for learning mathematics is given in Part 3. What I want to explore here is how writers have set out their statements about reasons. This is an important preliminary because many arguments concerning the justification of mathematics seem to rest simply upon a dichotomy between ideas of usefulness and *non*-usefulness.

If we allow all the examples that I have discussed so far to be collected under the notion of the usefulness of mathematics, then it seems that there is scarcely any other reason for learning the subject. Then, not only would applications of mathematics to a range of extra-mathematical situations be cases of the usefulness of mathematics. The use made of certain parts of mathematics within the whole discipline, and its 'use' as a means of training the mind would also fall under the same category. All of these reasons would be of one kind. Yet, as I have said, there are those who still want to suggest that there is some justification for learning mathematics which does not invoke the notion of usefulness.

One way of asserting that the value of mathematics does not lie in its usefulness, is to show that mathematics is not the means to any end, it is an end in itself. Another way of putting this has been to say either that mathematics has *intrinsic* value, or that it is worth learning for *its own sake*.[18] Precisely what this latter expression can mean will be considered later. What I want to do firstly is to see how writers have managed or failed to manage to present reasons of different kinds. This will then form a preface for a further enquiry into particular kinds of reason.

Is there a plurality in kinds of reason?

In Plato's *The Laws*, as we have seen, emphasis is placed on the utility of mathematics. *The Republic*, too, with its remark about the value of mathematics for the soldier indicates how mathematics is useful. But *The Republic* provides an early example of how theorists have wanted to mark out different *kinds* of reasons for learning mathematics. Not only must soldiers study mathematics for organising their armies, but *philosophers*, too, must study them for what appears to be a radically different end, namely, so that they can 'escape from this transient world to reality'. Now, if the study of mathematics does provide the wherewithal for 'escape', whatever this may mean, then we might want to say that this is just one more of its uses. But Plato and certain other writers who have given what appear to be related or modern versions of this purpose, want to suggest that studying mathematics for this end is not just another example of the utility of the subject.

Another interesting case arises from what is said about the value of mathematics by Vives, the Spanish tutor to Mary the daughter of Henry V111, who set out the case for mathematics in the first universities. He makes no mention of the clear cases of utility which I have been discussing here. Instead, for Vives, the purpose of learning mathematics was to, '... "display the sharpness of the mind" and to provide discipline for "flighty and restless intellects which are inclined to slackness and shrink from or will not support the toil of a continued effort" ' (See Howson, 1982 p.5). But he still believed that there was potential danger when the subject was pursued in such a way that it, '... "leads away from the things of life, and estranges men from the perception of what conduces to the common weal" ' (p.5).

This suggests that mathematics can act as a *vehicle* for performances of some kind and, thereby or otherwise, play a part in character building. Yet Vives' remarks came with a

[18] For a discussion on the distinction between intrinsic worth and things done for their own sake, see Wringe (1988) Ch.11

warning: engagement in mathematics could be counter productive if one became too absorbed in the subject. It is not clear whether this is a warning against the very purpose of mathematics, which Plato set out for the philosopher, but the similarity between what Vives was warning his readers about, and what Plato seemed to be welcoming in mathematics, is striking. Both thought that mathematics had a special power to lift the student away from the familiar world. Both articulate this in terms of a 'leading away, from 'perception'. Yet Plato and Vives have contrasting, if not diametrically opposed, beliefs about the value of such a transformation in the individual. Furthermore, for Plato the escape was reserved for the elite few who were picked out not to take a direct part in the 'common weal'. Why Plato could value so highly this 'leading away' will be discussed later. For the moment, all that I want to show is how utility is not the *only* reason which has been given by some theorists for pursuing mathematics.

Duality in reasons

Some important controversies follow from the observations just made. Indeed Howson uses Vives' remark not only to signal the possibility of *two*-fold reasons for learning mathematics, and to show how these are endemic in mathematics education, but also to show how crucial such an observation is in understanding the development of mathematics education in England. He writes:

> The *dual* aspects of mathematics, the practical and the contemplative, were clearly distinguished then by Vives, and he recognised the need healthily to reconcile the two. *The subsequent history of mathematics education in England is largely a chronicle, on the one hand, of how this problem was ignored – with the result that a bipartite system of mathematics education was effectively created – and, on the other hand, of how individual educators have constantly sought to effect a reconciliation.*[19](Howson, 1982 p.5)

Whether Vives has in fact clearly distinguished the particular dual aspects, to which Howson refers, is questionable. The 'practical' and the 'contemplative' seem to fit in rather more with what has been said so far about Plato's rationale. Moreover, whilst Vives does refer to the 'common weal' he does not imply that mathematics is instrumental in achieving this. But clearly for Howson, dualities, perhaps even *pluralities*, in reasons exist, and it will be my task here to identify them and examine them.

[19] My italics

When writers have wanted to justify mathematics both for its usefulness and for some other reason not characterised by utility, then they present what I shall call a *dualist* justification. For example, Griffiths and Howson (1974) make the following statement describing a duality in mathematics education:

> From the Egyptians onwards, mathematics has had *two aspects*. One of these is *practical*, as a help in commerce, farming, building and control of the environment; the other is *aesthetic*, as men have enjoyed the contemplation of numbers and geometrical forms, the discipline of controlled imaginative thought and the thrill of discovering new mathematical relationships. The Egyptian priests combined both aspects, the *mystical and the practical*, and their society honoured them highly for both.[20] (p.7)

The idea that mathematics is of '*help*', here, in achieving other goods is clearly an assertion that in this respect mathematics is useful, though the authors prefer to use the word 'practical'. Whether or not 'practical' and 'useful' are interchangeable will remain to be seen. What Griffiths and Howson must provide, if there is to be a distinction between the reasons which arise from the two aspects which they have identified, is something that is *non*-practical. But, they have not been scrupulous in conveying the exact meaning of this second aspect. Firstly, they refer to it as 'aesthetic'. Yet under their concept of aesthetic they curiously include the idea of 'disciplined thought' of some kind, and then in the next sentence 'aesthetic' is equated with 'mystical'. Later on, they again seem to identify 'the aesthetic aspect of mathematics' with 'mathematics as a mental discipline'. Now, whilst they may not intend to *equate* 'aesthetic' and 'mental discipline' (p.15), they certainly make a very strong connection between these two ideas. This is of some considerable importance, because the authors seem to think that the practical and the aesthetic are the key concepts underlying the reasons given for learning mathematics. As they say:

> Just as the mathematics had its two aspects – the practical and the aesthetic – so we shall see that at different times and in different places both aspects were used to justify the teaching of the subject. (pp. 8-9)

So it is of no small importance to clarify this notion of aesthetic, especially since amongst some writers what appears to be a further step is made when mathematics is

---

[20] My italics

compared with *fine art.* The nature and connections between aesthetics, art and mathematics will be explored in detail in the next section. At this point, however, it is worth noting that since at least the time of Kant, the idea of aesthetic has come to mean something non-purposeful, a form of disinterested enjoyment. So any aesthetic aspect of mathematics, if it exists, would seem to be a perfectly suitable candidate for a reason of a different kind from those which embody, or are examples of, usefulness.

Unitary justifications - some examples

Justifications based almost solely on what appear to be broadly aesthetic aspects of mathematics are not hard to find. Wells (1989) in providing an answer to his own question 'why do mathematics?' asks: 'What is attractive about mathematics? Wherein lies its magnetic quality? How does it hook your attention and draw you in?' It must be said that his use of 'attractive' and 'hook' do give the impression that he is addressing his question to those who are already devotees. His approach is hardly for those who are sceptical about the value of the subject and seek a justification for devoting time and energy on learning mathematics, rather than on other pursuits. Nevertheless, he does offer the basis of what seems to be a unitary justification for learning mathematics, referring to several aesthetic qualities:

> Mathematics has been variously described as mysterious, puzzling, surprising, weird and curious; elegant and beautiful; simple and complicated; displaying generality, unexpected connections, hidden depths, unity-in-variety; and last but by no means least, as extraordinarily powerful. (p.34)

I am not suggesting that Wells would deny that mathematics was useful, but his failure to stress this shows that a different set of reasons are the operative ones. In a similar way, the remarks of Whitcombe (1988), about the value of mathematics, are similarly unitary. Complaining that various national publications have in the recent past concentrated on 'the utilitarian aspects of mathematics', thereby impoverishing the mathematics curriculum, he writes that '...the wellsprings of mathematics are not utility and relevance, but creativity, imagination and an appreciation of the beauty of the subject.' It is questionable whether this is entirely true. Much depends upon what is meant by 'the wellsprings'. If it simply means the origins of a particular research project in mathematics then it is probably far from the truth. Mathematics has surely on occasions arisen from mere curiosity and fascination with a particular homespun problem or enquiry, but much of mathematics unquestionably arises from the seeking of solutions to pressing practical problems. Nevertheless, the view that mathematics has predominantly arisen from non-practical sources has been influential in education. In her manual for primary teachers Leibeck (1984) maintains a similar view with respect to the reasons for learning the subject. She writes:

> Some people may enjoy mathematics because it is useful. But it is far more likely that its appeal for us lies in the intellectual or aesthetic satisfaction that we derive from it. This is particularly true of children. (p.13)

So we certainly do appear to have the basis for different kinds of reasons in these writers.

Chapter 3 – The scope *of 'useful'*

Mental training and utility

Let us return to the two aspects of mathematics discussed by Griffiths and Howson from which the authors suggest that reasons of two broad kinds arise. (see above page 36) We have already noted that these authors identify non-practical reasons arising from aspects of mathematics variously referred to as 'aesthetic', 'mystical' and those which provide 'mental discipline'. But if mathematics provides mental discipline surely this is just another way in which it is practical. One difficulty in responding to this point, is that it is not clear precisely what aspect of mathematics is supposed to provide such discipline, and thus how it might function as a reason for learning mathematics. Sometimes what is meant is that mathematics embodies *order* of some kind and that in learning it one acquires a more ordered mind. So that mathematics, since it is the means of reaching such a mental state, is thereby useful. Others, however, have thought that this 'use' of mathematics is altogether different from the other uses of mathematics. Nevertheless one can easily feel rather torn on the matter.

Whitehead was one writer who did seem to think that the development of mental discipline was part of the utility of mathematics. When he gave a lecture on the mathematics curriculum earlier in this century he seemed at first to give the impression that mathematics was not in need of any justification. He remarked that it was not his task 'to defend mathematics as a subject for profound study', since, as he put it, 'it can very well take care of itself'. He took it for granted that the subject was of value; what he wanted was a modified curriculum. As things were, the curriculum was, he thought, 'too recondite', and so:

> ... the very reasons which make this science a delight to its students are reasons which obstruct its use as an educational instrument – namely, the boundless wealth of deductions from the interplay of general theorems, their complication, their apparent remoteness from the ideas from which the argument started, the variety of methods, and their pure abstract character which brings, as its gift, eternal truth.
>
> (Whitehead, 1929 p.118)

So the mathematics curriculum as it stood was unable to satisfactorily achieve its goal, as an 'educational instrument'. Thus despite Whitehead's not intending to provide a rational for studying mathematics, we do find one emerging when he remarks that 'this liability to reconditeness is the characteristic evil which is apt to destroy *the utility of* *mathematics* in liberal education'.[21] (p.117) What is interesting is how he connects, here, the seemingly disparate ideas of 'utility' and 'liberal education'. The idea of a liberally educated person has usually been opposed to the idea of one who is trained for a vocation. Moreover, those who try and suggest that mathematics is not justified by its usefulness are often those who wish to uphold a liberal view of education, which they believe an emphasis on the utility of mathematics undermines. But it seems that all Whitehead is saying here is that mathematics can be the means for which the liberally educated person is the end. Consistently with his view of liberal education, he elaborates his reasons for teaching mathematics a little further saying that, '...elementary mathematics rightly conceived would give just that *philosophical discipline* of which the ordinary mind is capable' [22] (p.122). His later use of the 'utility of mathematics' gains a little more sense when he speaks of mathematics as, '...the chief *instrument* for discipline in logical method' [23] (p.127). If Whitehead is suggesting that mathematics forms some kind of mental training or discipline, as many have done, he is also clearly suggesting that this is one way in which mathematics is useful. But as we have seen other writers have wanted to distinguish the usefulness of mathematics from its value as a form of mental training, so that usefulness and mental training provide different kinds of reasons. What appears to be a dualist justification can, then, collapse into a unitary one especially if we broaden the scope of 'usefulness'.

Sometimes as we have seen in the account of Griffiths and Howson (see page 36) the view that mathematics is useful is expressed more narrowly by ascribing '*practical*' value to the subject. So that even if we agree with Whitehead that mental training is one of the *uses* of mathematics, it is still not clear whether this is quite the same as saying that mathematics is thereby of *practical* value. It depends upon whether there is an important distinction between 'useful' and 'practical' or whether they are interchangeable. One could argue that the notion of what is practical implies that the learner is always aware of the end for which he or she is using mathematics. Whereas, the mental training which mathematics is supposed to provide is not something which the learner, *himself or herself*, uses mathematics to achieve, although the educator might apply it in this way. Moreover, mental training is more like something, which *happens* to him or her in learning mathematics rather than something which either *does*

---

[21] My italics
[22] My italics
[23] My italics

with mathematics. Finally, we might want to argue that mental training is not something which is so immediate as the purely practical ends of mathematics can be. But writers have seldom, if at all, made such a distinction between different kinds of *uses* of mathematics, so the way seems open for different views on the matter. Whether or not these conditions can be used to distinguish the useful from the purely practical will remain to be seen, after a consideration of the different *kinds* of opportunities for mental training have been discussed in Part 3. But one example can be given of a theorist who did distinguish the useful from the practical in order to separate off the mental training outcome from others, thus providing different kinds of reasons.

I suggested earlier that there was indeed a duality in Plato's justification for learning mathematics in *The Republic*. A similar duality, in more modern dress, is set out by Sir Joshua Fitch, an inspector of training colleges. In his *Lectures on Teaching* given in 1880, Fitch (1902) writes:

> There are two conceivable objects in teaching any subject. (1) Because the thing taught is necessary, or useful, and may be turned to practical account, or (2) because the incidental effect of teaching it is to bring into play and exercise certain powers and capabilities, and so to serve a real educational purpose...of Arithmetic we may safely say at the outset, that if rightly taught, it is well calculated to fulfil both purposes. (p.286)

Fitch's account is interesting, because although he does want to invoke the notion of 'practical', this is made to follow from the usefulness of mathematics. But, then for his second 'object' to be different in kind cannot therefore also follow from the usefulness of mathematics. This second object is the mental training aspect which, as I have remarked before, is still not unquestionably free from being a species of the usefulness of mathematics in the hands of other writers. Of course, for those who want to stress the usefulness of mathematics, no further refinement is necessary. Indeed, it may seem to them to be mere pedantry to pursue the matter. But as doubt has been cast on the 'usefulness' of mathematics from time to time, it is essential to provide rather more analysis of the supposed 'non-useful' reasons.[24]

[24] At least one writer has recently suggested that the justification for teaching mathematics on the basis of its usefulness, in one its senses at least, is a *myth*. (See Andrews, 1998 and also my response in Huckstep, 1999).

The denial of duality in justifications

My point in questioning whether or not some reasons for learning mathematics really are free from the all taint of usefulness becomes much clearer, I think, when we consider a thoroughgoing unitary rationale. John Perry (in Griffiths & Howson 1974) wrote at the turn of the twentieth century that:

> 'The study of mathematics began because it was useful, continues because it is useful, and is valuable to the world because of the usefulness of its results, while the mathematicians, who determine what the teacher shall do, hold that the subject should be studied for its own sake' (p.17)

In this assertion Perry gives the impression that he is ruling out a particular reason for learning mathematics, namely pursuing it *for its own sake*. This reason, mentioned earlier, seems to be behind what educationists like Wells (see above page 37) give as reasons for 'doing' mathematics. However, Perry's denial is only made at the cost of his extending the concept of the usefulness beyond reasonable bounds. In the Report, which bears Perry's name, it was claimed that mathematics is useful in no less than eight ways, many of which sound rather quaint. They are:

> (1) In producing the higher emotions and giving mental pleasure. Hitherto neglected in teaching almost all boys.
>
> (2) (a) In brain development. (b) In producing logical ways of thinking. Hitherto neglected in teaching most boys.
>
> (3) In the aid given to mathematical weapons in the study of physical science. Hitherto neglected in teaching almost all boys.
>
> (4) In passing examinations. The only form that has not been neglected. The only form recognised by teachers.
>
> (5) In giving men mental tools as easy to use as their legs or arms; enabling them to go on with their education (developing their souls and brains) throughout their lives, utilising for this purpose all their experience. This is exactly analogous with the power to educate one's self through the fondness of reading.
>
> (6) Perhaps included in (5): in teaching a man the importance of thinking things out for himself and so delivering him from the present dreadful yoke of authority and convincing him that, whether he obeys or commands, he is one of the highest of beings. This is usually left to other than mathematical studies.

> (7) In making men in any profession of applied science feel that they know the principles on which it is founded and according to which it is developed.
>
> (8) In giving to acute philosophical minds a logical counsel of perfection altogether charming and satisfying, and so preventing their attempting to develop any philosophical subject from the purely abstract view, because the absurdity of such an attempt has become obvious.
>
> (Quoted in Griffiths & Howson 1974 pp.17-18)

The difference between Perry and Fitch is striking. Whereas Fitch wanted to distinguish two kinds of reason, Perry wants to draw almost every possible reason he can think of under one broad kind. So that in his list we do find something resembling mental training appearing under reason (2) thus gathering it under the umbrella of a very broad notion of usefulness. Thus his view is in contrast to both Plato and Fitch who both implied that usefulness and mental training were two distinct reasons for studying mathematics. What is even more remarkable is Perry's reference to the production of 'the higher emotions and giving mental pleasure' (No. 1 in his list). This reference to 'pleasure' seems to include in one fell swoop most of the value identified by those who want to justify mathematics for its own sake. Indeed, it is often supposed that the *meaning* of doing something for its own sake is doing something simply for pleasure rather than as a means to some other end.[25] But this is precisely what Perry's own justification, with its exclusive insistence on usefulness, is supposed to reject. I have already shown that enlarging the scope of the usefulness of mathematics to include mental training is by no means uncontroversial. I now want to take a closer look at what is meant by 'doing something for its own sake' to see what bearing this has on unitary justifications of the kind which Perry has set out.

Doing something for its own sake

Champlin (1987) in getting clear about the idea of doing something for its own sake sets out to attack a view which he calls 'isolationism'. He applies this idea to a range of reflexive forms of verbs like 'self-raising' and 'self-correcting'. Firstly, take the case of what is called 'self-raising' flour. If such flour undergoes a chemical analysis it will be found to contain baking powder. So it is not *self*-raising after all. Similarly if we look

[25] I make this point notwithstanding Aristotle's remark that pleasure is in a sense instrumental in

under the dashboard of a car we find that what were supposed to be 'self-correcting' indicators are really corrected by a cam. But to deny that the flour is self-raising and the indicators are self-correcting would be, according to Champlin, cases of isolationism. This is because we are looking in the wrong place for our answers. The flour is *self*-raising because the *cook* does not have to add further ingredients. The indicators are *self*-correcting because the *driver* does not need to correct them. The use of 'self' in these examples acquire their meaning not from within the substance of the flour and within the mechanism of the indicators. Their meaning is acquired, Champlin argues, from the background in which they are set. If we look in the flour and under the dashboard we are looking in the wrong place. We need to consider the context in which they are used.

In philosophy Champlin's main target is Kant, who he says, using his earlier analogy, was 'trying to do moral philosophy with, so to speak, his head stuck under the psychological dashboard'. (p.37) This is because for Kant, he argues:

> To pursue virtue for virtue's sake, nothing must count with you but virtue alone. All else must be excluded – pleasure in the happiness you bring to others, contentment, peace of mind, good nature, fellow-feeling, a sunny disposition. (p.35)

With this in mind, Champlin gives another example in this Kantian style. Here a person has chosen some music for its own sake, yet adds that the music is beautiful and gives him (or her) great pleasure. S/he is immediately rounded on by someone who objects that the music was supposed to have been chosen for its own sake and it cannot thus be for the beauty and the pleasure it gives.

To this Champlin replies:

> But this, is, surely, quite mad. Somehow, in stripping the music bare, as we do when we exclude its power to evoke associated memories and to create nostalgia, in our desire to get at the music all by itself, we push things too far and end up having excluded something we need to keep in order to make sense of what it is to choose a piece of music for its own sake. (p.36)

In the context of learning for its own sake, which is the area which we are particularly concerned with, Champlin quotes from Newman's *The Idea of a University*, and finds

---

promoting happiness. (Ethics, 1953 edn pp. 36-7) something much more acceptable. What Newman does there, he argues, is not to strip knowledge of any possible ends, but to separate out and contrast *particular* ends. He contrasts, that is,

> ...the pursuit of knowledge for its own sake with the pursuit of knowledge for utilitarian ends – a liberal education with commercial, professional and vocational education. (p.40)

Comparing Newman's position with Kant's he remarks:

> How different is Newman's account from the one Kant would have given, who, if he had to depict the university academic pursuing his subject for its own sake, might have imagined a scholar on whom his subject had gone dead but who chillingly persevered with his researches, his reading and his lecturing, although he could take no satisfaction or enjoyment from them. To omit the thirst for knowledge, the intellectual enthusiasm, in order to guarantee that it is knowledge alone that is pursued, is like arguing that if the driver has to use his own muscle power to straighten the wheel before the self-cancelling indicator works, then it is not a truly self-cancelling device... (pp. 40-41)

He concludes by remarking that:

> The line between what is internal and what is external to that which is done for its own sake needs to be drawn in a much more relaxed way: sufficiently relaxed to permit the line to be drawn in different places for different purposes. (p.46)

What reply then could Perry, or indeed anyone who is prepared to enlarge the scope of 'usefulness' in his way, give to the charge that the scope of 'usefulness' now includes the very thing which the whole point of the enlargement was supposed to prevent being acceptable? He could deny that item (1) on his list did entail doing mathematics for its own sake, though it is difficult to see how this reply can be made without adopting an isolationist position of the kind which Champlin has convincingly shown to be untenable. He might, however, invoke another sense of 'doing mathematics for its own sake' where it is the sake of the *mathematics* which counts rather than simply the individual's enjoyment.

A different sense, of pursuing an activity for its own sake, along these lines, is marked out well by what Nagel (1979) calls 'perfectionist ends', those ends which are characterised by 'the intrinsic value of certain achievements or creations, *apart from* *their value to individuals who experience or use them'*.[26] The list of contexts in which such ends are possible is broad and Nagel elaborates upon some helpful examples:

> Examples are provided by the intrinsic value of scientific discovery, of artistic creation, of space exploration, perhaps. These pursuits do of course serve the interests of the individuals directly involved in them, and of certain spectators. But typically the pursuit of such ends is not justified solely in terms of those interests. They are thought to have an intrinsic value, so that it is important to achieve fundamental advances, for example, in *mathematics* or astronomy even if very few people come to understand them and they have no practical effects. The mere existence of such understanding, somewhere in the species, is regarded by many as worth substantial sacrifices.[27] (pp. 129-130)

Nagel's account of perfectionist ends differs from the sense of doing something for its own sake, outlined by Champlin. For Champlin, no other explanation beyond an individual's pleasure, enjoyment or, what he calls at one point 'a thirst for knowledge', is necessary to indicate that something is being pursued for its own sake. The pursuit of perfectionist ends, on the other hand, is likely to bring at least some enjoyment, but this is not essential. It is the good of the activity, rather than the good of the individual who undertakes that activity, which is the decisive factor.

An interesting point now emerges: in pursuing a discipline for *its* own good, usefulness, in one of its senses described earlier, reappears. This is because certain things now become useful in enabling the discipline to flourish. When G.H. Hardy (1967), remarked, that '...very little of mathematics is useful practically, and that... little is comparatively dull', he still supposed that mathematics was useful, but that it was useful in the slightly special way in which I mentioned earlier. Certain mathematical *results* are useful, because they are the means of achieving ends *within* mathematics. So that parts of mathematics may have value because they are, as Hardy puts it, 'serious' or 'significant':

> The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the *significance* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. (p.89)

---

[26] My italics

[27] My italics. One of the best examples in recent times is the work of Wiles in solving Fermat's Last Theorem.

In saying this, Hardy embraces a view that values the pursuit of mathematics for the flourishing of the subject itself, independently of any flourishing of the individual. This view has its ancestry in writers like Alexander Malcolm, who urged the 'doing of mathematics for its own sake' in *both* of the senses I have discussed here. Although utility for posterity is not far from his mind, Malcolm (1730) does, in the following, anticipate Nagel's perfectionist ends:

> Though there be many truths discovered in the Theory of Arithmetic of which there has been no use or application yet found, there is no reason why those things should be neglected or kept out of the system; for they are still part of the Science, which we ought to enlarge more and more, as far as we can: one age may find the use of the theory which a former has invented.

But he also urges engagement in mathematics for enjoyment on behalf of the individual as he continues:

> I shall but add this one thing more, *viz*. That though many things in the Science of Numbers were supposed to be of no particular use in human affairs, yet as the mind of man is made for knowledge and contemplation, and the pleasure arising from the perception of beauty and order in other things, is allowed to be worthy of rational natures; the contemplation of the surprising connections, the beautiful order and harmony of relations and dependencies found among numbers, is not less reasonable...

If the engagement in mathematics for its own sake, in either of the two senses which I have discussed, were largely a matter for professional mathematicians, like Hardy, then it might be supposed that its relevance as a justification within educational contexts is questionable. But Elliott (1974) has shown that any individual can have these kinds of attitude towards a subject like mathematics during the stages in which it is being *learnt*. Elliott gives us some considerable insight into the relationship which an individual can have towards various aspects of a subject. One important point which he makes is that this relationship is not in a steady state. This he argues by drawing an analogy between the love of a subject and the love of a person. In a way reminiscent of Wells, (see above page 37), Elliott attributes the initial reasons for wanting to learn a subject to its *attractiveness*. One is, as it were, 'drawn into' a subject and then becomes 'good' at it, better at it than other subjects. It begins to consume more and more time and it must be realised that there are standards to meet which involve considerable work, sometimes bordering on toil. But whilst love for the subject remains, Elliott explains how a change of attitude can take place within the individual:

> Pleasures do not come easily now, but he finds a fulfilment in trying to satisfy the demands his subject makes upon him. He has become devoted to its discipline, and feels at times that he has been enlisted in its service. His relationship to his subject now bears an analogy to courtly love, as celebrated by such authors as Chaucer, Gower and Chrétien de Troyes, a love which makes, extreme, even cruel, demands and offers no guarantee of pleasure. (p.136)

But indefinite lack of pleasure does not typify love of a subject. In time, the individual, Elliott adds, '...finds himself experiencing the old delight again, as if nothing had ever come between him and the objects he is studying'. Yet, this pleasure does not necessarily remain. It typically gives way to toil once again. This love is characterised by the 'cyclic manner' in which delights are obscured by toil and then delights emerge again.[28]

The pursuit of a subject in the face of there being no 'guarantees of pleasure', which Elliot outlines has what Nagel has called 'perfectionist value' and it thus does not always result in what Perry called 'mental pleasure'. So, to return to the question of how far theorists like Perry can extend the scope of 'usefulness' to rule out different *kinds* of reasons for learning mathematics, it seems that the only effect that such an enlargement of 'useful' can have is of ruling out the pursuit of mathematics for perfectionist ends. But whilst these ends might appear to be of little significance in education, Elliott's account shows not only that they are involved in the learning of any subject which one comes to love, but also how they seem to go hand in hand with the pursuit of enjoyment. We must conclude, therefore, that attempts such as that made by Perry to reduce all kinds of reasons for learning mathematics to one kind – those which may be described in terms of usefulness – cannot be sustained.[29]

---

[28] Not only is there a fluctuation between enjoyment and toil, in learning. A similar fluctuation can take place between wanting to enlarge the subject for its own sake and because it is from time to time found to be of use, even though in this respect, too, there are no guarantees.

[29] Perry's over-zealous attempt to stretch the scope of usefulness beyond helpfulness was not an isolated case. Davis and Hersh (1980), perhaps with a touch of irony, also show the same disregard for distinctions in kinds of usefulness, when they remark:
'A pedagogue particularly of the classical variety – might tell us that mathematics is useful in that it teaches us how to think and reason with precision. An architect or sculptor – again of the classical sort – might tell us that mathematics is useful because it leads to the perception and creation of visual beauty. A philosopher might tell us that mathematics is useful insofar as it enables him to escape from the realities of day-to-day living. A teacher might say that mathematics is useful because it provides him with bread and butter. A book publisher knows that mathematics is useful for it enables him to sell many textbooks. An astronomer or physicist will say that mathematics is useful to him because mathematics is the language of science. A civil engineer will assert that mathematics enables him to build a bridge expeditiously. A mathematician will say that within mathematics itself, a body of mathematics is useful when it can be applied to another body of mathematics subjects'. (pp. 79-80)

The breadth of usefulness explained – Mathematics for *communication*

Notwithstanding the extravagant scope which Perry allowed 'useful' to have, for Griffiths and Howson the plea for utility culminating in the report, marked a watershed in the development of British mathematical education, which had for so long resisted a practical mathematical curriculum. But the authors show, too, that the overriding demand for utility later became somewhat tempered. For example, they refer to the Dainton Committee who in 1968 did provide different kinds of reasons for justifying mathematics education. These were:

> 1. Mathematics as a means of communicating quantifiable ideas.
>
> 2. Mathematics as a training for discipline of thought and for logical reasoning.
>
> 3. Mathematics as a tool in activities arising from the developing needs of engineering, technology, science, organisation, economics, sociology, etc.
>
> 4. Mathematics as a study in itself, where development of new techniques and concepts can have economic consequences akin to those flowing from scientific research and development.
>
> (Quoted in Griffiths and Howson, 1974 pp. 22-23)

Even here, there is still a sense in which the first three of these purposes might still be regarded as species of usefulness. But, as I remarked earlier, and will reconsider in Part 3, it is not universally agreed that mental training is yet another example of the usefulness of mathematics. Yet, with the inclusion of item 4, slightly ambivalent as it is, something approximating to the perfectionist value discussed above has been introduced. It is, however, interesting to note that explicit mention of mathematics as a source of pleasure, mental or otherwise, is not made.

More recently, *Mathematics counts*, from the Cockcroft Committee, acknowledges the breadth of scope in the notion of usefulness. Thus it subsumes reasons strikingly similar to those in item (3) of the Dainton list under one similar in description to that given in item (1) which it treats as the *principal reason* for teaching mathematics to children:

> The usefulness of mathematics is perceived in different ways. For many it is seen in terms of arithmetic skills which are needed for use at home or in the office or workshop; some see mathematics as the basis of scientific development and modern technology; some emphasise the increasing use of mathematical techniques as a management tool in commerce and industry...we believe that all these perceptions of the usefulness of mathematics arise from the fact that *mathematics provides a means of communication which is powerful, concise and unambiguous.*[30] (Cockcroft et al (1982) p.1)

All of what is deemed useful in mathematics is drawn out and made to follow from this one source – mathematics as a special means of communication - which it treats as the *principal reason* for teaching mathematics to children. However, what the report calls 'the second important reason' is curiously just a filling out and an extension of some of the useful aspects of mathematics already mentioned:

> A second important reason for teaching mathematics must be its importance and usefulness [sic] in many other fields. It is fundamental to the study of physical science and of engineering of all kinds. It is increasingly being used in medicine and the biological sciences, in geography and economics, in business and management studies. It is essential to the operations of industry and commerce in both office and workshop. (p.2)

So far then the report remains unitary with respect to kinds of reasons given for teaching mathematics to children. However, two other kinds of reasons do follow. The first of these concerns what I have been referring to as mental training. In the report, however, unlike some of the earlier views I have briefly reviewed, this reason is fairly muted:

> It is often suggested that mathematics should be studied in order to develop powers of logical thinking, accuracy and spatial awareness. The study of mathematics can certainly contribute to these ends but the extent to which it does so depends on the way in which mathematics is taught. Nor is its contribution unique; many other activities and the study of a number of other subjects can develop these powers as well. We therefore believe that the need to develop these powers does not in itself constitute a sufficient reason for studying mathematics rather than other things. However, teachers should be aware of the contributions which mathematics can make. (p.2)

---

[30] My italics. Compare this remark with item 1. on the Dainton list.

We have already seen how the mental training reason has been subsumed under an extended notion of usefulness. Nevertheless, the Cockcroft Report does, at the very least, embody a dualist account on the strength of the final reason which it gives for teaching mathematics. Here mathematics is seen as having intrinsic worth as a form of *entertainment*:

> The inherent interest of mathematics and the appeal which it can have for many children and adults provide yet another reason for teaching mathematics in schools. The fact that 'puzzle corners' of various kinds appear in so many papers and periodicals testifies to the fact that the appeal of relatively elementary problems and puzzles is widespread; attempts to solve them can provide both enjoyment and also, in many cases, lead to increased mathematical understanding. (p.2)

We have seen then how a consideration of some categories can help us begin to analyse the reasons that have been given for justifying mathematics in the curriculum. In particular, there is the category of usefulness, which it seems can either be interpreted in a fairly narrow way, often then termed 'practical', in which the ends are direct, and immediate and consciously sought after, or 'useful' can be interpreted in a more thoroughgoing way. The narrow sense we have seen sometimes allows mental training to become a different kind of valuable goal of mathematics. Emphasising the intrinsic properties of mathematics, too, allows another category to emerge in justifications which writers have set forth. These as we have seen are often expressed in terms of aesthetic properties or are more generally regarded as the entertainment value of mathematics. Mathematics is then justified as a form of enjoyment of some kind. Those who do not want to argue that mathematics is solely justified by its usefulness, as I have pointed out, appeal to some of these other sources of mathematical value.

Summary and Conclusion to Part 1

Counting, measuring and the numerous examples of mathematical models, allow us at the very least, to determine quantities *indirectly*. In this respect mathematics may be regarded as a useful *tool*. But the concept of 'use' is not always restricted to just those ends for which mathematics may be regarded as a tool to determine quantities, both in everyday life and within other subjects of the curriculum. Mathematical results can also be of use within the whole discipline of mathematics. Moreover, from ancient times it has also been supposed that mathematics has a special kind of educational value in improving or training the mind in some more indeterminate way. Furthermore it has also seemed to be an activity which can bring much enjoyment to its participants – a form of entertainment. Whether or not the provision of some kind of mental training and entertainment by mathematics are just two more ways in which mathematics is useful has been central to my discussion. The question of whether there are different *kinds* of reasons for learning mathematics depends upon whether the notion of usefulness in mathematics can helpfully be extended in scope to encompass these other purposes of mathematics.

A review of the rationales given for mathematics from the time of Plato until the beginning of the 1980s shows that what is regarded as useful varies considerably. Sometimes what one writer sees as *two kinds* of reasons another sees as only one. Even amongst those who seem to think that there is only one reason for learning mathematics, there is disagreement. However, much depends upon notions such as 'useful' and 'practical'.

Partly in order to give 'useful' some force in justification, rather than allowing it to be mere rhetoric, I have tried to rescue both the mental training and the entertainment end from being absorbed into the notion of usefulness.[31] In particular, I have argued that 'doing mathematics for its own sake', in one of its senses, is meaningless if the pleasure for which it is undertaken is regarded as just one more end for the which the activity is a means. In another of its senses, 'doing mathematics for its own sake' need not involve pleasure but then it is for the flourishing of the discipline that such activity is primarily undertaken. Both cases provide reasons for studying mathematics which even the most thoroughgoing notion of 'usefulness' could not eliminate. Thus I have argued that there are not only different uses of mathematics, which provide one kind of reason, but different kinds of reasons for learning the subject. *How fit* mathematics is for certain of these purposes will be the subject of Parts 2 and 3.

[31] In this respect I concur with the Cockcroft Report *Mathematics Counts*.

Interlude

Beyond Utility – The legacy of Pythagoras.

We have seen in Part 1, that the learning of mathematics has been continually and convincingly justified in terms of its usefulness by many theorists. Indeed some theorists seem to think that this is the *only* way in which it should be justified (Perry in Griffiths and Howson 1974, see page 42), whilst others (Davis and Hersh, 1980, see page 48) show how every reason may be phrased in terms of usefulness. Nevertheless, there have always been theorist for whom reasons for learning mathematics can be found which do not rest upon utility. Such reasons I have traced back to Plato, not only because his rationale for mathematical education in *The Republic* is fairly detailed, but more importantly because his non-utility reasons are valued *above* the utility reason. A similar emphasis of utility over non-utility is clear in the rationales given by many other theorists too, and this is why an analysis of different kinds of reasons is so important in my enquiry. In Part 1, one of my tasks was to separate out the varieties of 'usefulness' and to try and hold on to the distinction that some theorists seem to have been destroying between the useful and the non-useful. This has meant that I have had to work with fairly vague expressions like 'mere utility' 'immediate utility' and 'practical', the meaning of which I have tried to distinguish from the 'usefulness' of mathematics when it is being used by the educator rather than by the pupil.

The distinction between the two interpretations of 'doing something for its own sake' was more satisfactory in bringing out a basis for different kinds of reason. Recall that one of these interpretations – doing something for the sake of *mathematics* - leads back to the idea of enlarging the subject and hence of partly pursuing that which is useful *within* mathematics. On the other interpretation of the expression 'doing something for its own sake', where enjoyment of some kind is essential, I have shown that any attempt to allow usefulness to be introduced automatically invalidates the purpose of that expression. Through this route in Part 1, I hope to have allowed non-utility reasons to have some clear sense. Without this sense the aspects discussed in Parts 2 and 3 are just sub-species of utility, and I want to argue that rationales for learning mathematics can extend *beyond* utility. But if the arguments from Part 1 are still not convincing I hope to show that broadening the historical perspective will provide more insight into reasons which are remote from utility. Moreover, it seems to me that insight into the origins of both of the topics in Parts 2 and 3 can be gained by going back further into history. Thus the purpose of the following interlude is to support the main conclusions of Part 1 and to set the scene for the following Parts.

Pythagoras Mathematics, and the Purification of the soul

One of the earliest sources of our belief that mathematics is more than simply a useful tool derives from Pythagoras. Nothing of what he wrote survives; we have to rely on the writings of others, in particular Aristotle. However, the general picture that emerges from those who wrote about him, is that like his predecessors and contemporaries, he sought an explanation of the nature of the world, in terms of what it was made of. But unlike them he did not conclude that it was essentially material. Pythagoras is characterised as saying that the world is essentially *number*. So impressed was he by such observations that musical notes are a function of the ratio between the lengths of vibrating strings, that he seemed to have felt that the ratios themselves were a more fundamental explanation of the musical sounds than the vibrating material strings themselves. This point he seems to have generalised, so that numbers are behind everything, and thus are bestowed with something of a godlike character. It is therefore only a small step to make to suppose that contemplating numbers is an essential way of becoming more godlike or pure.

Pythagoras and his followers founded a religious movement which embodied the idea of salvation. Human beings, they believed, have for some reason fallen from grace. When they die they are reincarnated, only to die and become reincarnated again in an everlasting process unless they are purified. For them purification is achieved not only at the lowest level by acknowledging certain taboos, but also at the highest level by what Koestler (1959) calls '…contemplating the essence of all reality, the harmony of forms, the dance of numbers'. (p.37)

This remarkable theory of the connection between mathematics and human well-being persisted in some respects in the philosophy of Plato and continued through the Middle Ages. Even in recent times, the *contemplative* aspects of mathematics has continually given rise to the view that mathematics can either nourish the soul or somehow train the mind. It nourishes and trains as one contemplates the order of things which mathematics seems to reveal, or sets the mind reflecting and thus acts as a 'kick-start' for further thought about a number of essential matters. For some, mathematics viewed in this way has become a species of *fine art* alongside music, paintings, sculpture and literature.[32]

Pythagoras was supposed to have been the originator of what were known as 'the three lives' by analogy with those who attended the Olympic Games. In ascending order of value these were those who came to trade, those who came to compete and those who came to watch. Those who 'watch' rather than 'act' in life were special. Warner (1958) tells us that with Pythagoras:

> This idea of the dignity, even the sanctity, of the contemplative or philosophical life is a new idea and was to be further developed. It is a moral idea and implies the consideration of man's duty with regard to his soul and the soul of others. (p.20)

Pythagoras and his followers were ardent students of mathematics and through their discoveries and reflections they came to the conclusion that 'everything is made of numbers'.[33] This view in one of its interpretations persists to the present day in the importance of measurement in science and the pursuit of mathematical laws of the universe. But as Warner points out:

> To the Pythagoreans, excited, as was natural enough, by their discoveries, there was something sacred in numbers themselves. Numbers and their arrangements expressed quality as well as quantity. (p.22)

There are two contrasting aspects to the Greek soul, one which is rash, wild and full of fury, the part often characterised in Greek tragedy, and the other which has a calm rational aspect. The soul, therefore, by its very nature was in need of some kind of remedial treatment and the notion of goodness as 'purification' became an important idea. Two sources which emphasise the ideal of purification, in the ancient world, are discussed by Passmore (1970). Firstly, he cites the poem 'Purification', written in the 5th-century BC by Empedocles, about which he writes:

---

[32] This is not the only reason implicit in the claims of those who try to persuade us that mathematics is a member of the fine arts. Many writers stress that it is the *creative* element in mathematics which warrant its being viewed in this way.

[33] As Aristotle succinctly says of the Pythagoreans: '...they thought that the principles of mathematical entities were the principles of all entities...' (Aristotle, 1998 edn).

> Man, according to Empedocles, is a demi-god, who, at the beginning of human history, committed a crime, involving the shedding of blood, for which all men since have had to pay the penalty. Banished from their proper home among the gods, men must live, as a consequence of their guilt, in cycles of reincarnation, life after life defiled by sin until, by the exercise of purifying virtues, they finally return to earth, like Empedocles himself, as 'prophets, bards, doctors or statesmen'. Then at last they escape from the cycle: 'I go about among you all,' Empedocles therefore writes, 'an immortal god, mortal no more.' (pp.36-37)

Secondly, he shows how purification was central to Orphic-like religions, a set of beliefs circulating in 5th century BC Athens. In the most prominent of these religions Passmore claims that the human is described as being '…latently immortal and divine; his latent divinity can be made manifest only if he purifies himself by ritualistic and ascetic practices' (p.37). It is not clear how or whether or not the association between Pythagoras and his followers and the necessity of purification is linked to either of these two sources, all that can be said is that both were contemporary beliefs expressed at the time.

But there seem to be two sides to the Pythagoreans. One side of them revealed a commitment to a rather strange mystery religion which forbade certain strange practices. On the other side they were portrayed as accomplished mathematicians and astronomers. But the notion of purification remains central in this more rational side of their being, not it must be admitted by the strange practices just mentioned, but, according to Passmore by a set of ideas namely 'the idea of contemplation, the idea of order or harmony, and the idea of purification by wisdom' (p.38). This latter notion of wisdom acquires a new meaning with Pythagoras. It no longer means practical ability as it did with Homer. It is now firmly attached to the ideas of contemplation. Referring at last to the role of mathematics in all this Passmore sums up as follows:

> The contemplation the Pythagoreans thus extolled is contemplation of the order of the universe, and especially of its *mathematical order*. With the help of such contemplation, the soul identifying itself with the order of the Universe could purify and perfect itself and thus emerge from the cycle of transmigration.[34] (pp.38-39)

[34] My italics

Plato and St.Augustine

Here, then, is the idea that mathematics is not simply useful but is more precisely 'good for us'. We have the potential to be godlike but have fallen from grace. To restore our true nature we must undergo a programme of contemplation. That which we must contemplate is the order of the universe. Since the universe is made of numbers, the order which we must contemplate is a mathematical order. So that engagement in mathematics is perhaps the first precursor of the idea that mathematics can train the mind. This idea is spelt out more clearly in Plato where its moral value, rather than simply its Godlike one, is stressed. For Plato, as we shall see, the soul by its very nature is in a potential state of conflict the resolution of which was supposed to be achieved by the pursuit of reason. Plato believed that mathematics was a route to such reason. Although Plato, like Pythagoras, still regarded the pursuit of harmony in the soul as a form of purification some modern commentators have preferred to describe this process as 'mental training', and thus the purification of the soul begins to acquire a modern dress. Not all kinds of mental training, however, may be traced back to the contemplative source of Pythagoras, as I shall show when I discuss the varieties of this particular rationale for mathematics. However, the Pythagorean contemplative aspect is not only one of the earliest origins of what has become mental training, it is also one of the most persistent. Even if it is not always present in theories of mental training, it is also evident in the other main aspect which I shall be discussing in connection with mathematics in Part 2, namely, art and aesthetics.

The link between contemplation, the purification of the soul, or at least the *nourishing* of the soul, is perhaps easier to make with the pursuit of art and aesthetics.[35] Moreover, the absence of utility is at its most convincing in this sphere. One again we can, I think, trace some of these connections between art and mathematics back to Pythagorean doctrines. The link is made partly by Plato, but more specifically by St. Augustine in the Middle Ages where we have elaborate statements of how mathematics is the basis of art and beauty. It is thus towards a consideration of the connections between mathematics and art, and how these might be relevant to a non-utility rationale for the learning of mathematics, to which we now turn.

---

[35] I am deliberately keeping these ideas of art and aesthetics separate for the moment.

Part 2 - Art and aesthetics in Mathematics Education

Chapter 4 – Mathematics and Art - some connections

Mathematics is a deeply philosophical pursuit and, of all the sciences is closest to the arts (Observer 28th September 1997)

Justification by Association

It has been customary to distinguish between the two human enterprises of art and science. Thus Aristotle claims '...scientific knowledge is of things that are never other than they are...the business of every art is to bring something into existence...' (Aristotle, 1953 edn pp. 174-175). Similarly, but much later, Descartes drew his own distinction by asserting that the sciences '...entirely consist in the cognitive exercise of the mind...' whilst the arts '...depend upon an exercise and disposition of the body'. (Descartes, 1969 edn pp. 35-36) As it stands these two views do not conflict. Indeed they may be combined. The sciences then become the pursuit of knowledge of what *is*, by exercising the mind, whilst art is the use of the body to bring something into existence.[36]

Suppose then, that we can distinguish between the sciences and the arts, and that provisionally the traditional manner in which this distinction is made is roughly correct. Suppose, too, that we rightly value the studying of *both* the sciences and the arts, that is to say that we value studying those disciplines which pursue knowledge of what *is*, by exercising the mind, and those in which use of the body brings something into existence. Then some of the subjects on the curriculum do not seem to be distinctly art or science. Few of the sciences, if any, seem to reach knowledge, *of what is*, purely by exercise of the mind since experimentation is necessarily involved in the process of its attainment. Also, as far as the arts are concerned the use of the body is not involved, except in a contingent way, in bringing literature, for example, into existence since it can surely be conceived in the mind.

---

[36] The notions of art and science have, of course, undergone some changes through the years, and the boundaries between them have become somewhat blurred. But I want to try and explore the traditional concept first of all. It is worth pointing out that these categories still feature in the curriculum, and most of our degrees still bear such names as Bachelor (Master) of Arts or Bachelor (Master) of Science. Many of the subjects on the school curriculum, too, are categorised as 'arts' (sometimes humanities) or 'sciences' and discussions surrounding pupils' choices of courses often include considerations of whether one should take an *art* or a *science*, so that these descriptions have acquired some importance.

Suppose we relax the condition by which the outcomes of the sciences and the arts are attained and simply focus on those outcomes. Then we are left with the situation that amongst the subjects on the curriculum are those which are concerned with knowledge of what *is*, and those which are concerned with what is brought into existence. The difficulty now is to show how these are always distinguishable. Is it not possible to regard some subjects as somehow bringing *knowledge* into existence? Some answers to this question take us into epistemological questions which I shall not discuss here. All I want to show is that whilst we mix and match pithy definitions of the sciences and the arts, the dividing line between them is sometimes vague. Yet we value *both.* For most purposes perhaps the vagueness is of no importance. But it does become important when use is made of science or art to persuade us that a particular subject is worth studying. This is especially true of mathematics.

Mathematics is not straightforwardly either an art or a science. But whilst it would be futile to try and *show* that, say, physics is a science, or that sculpture is an art, it is quite common to find writers suggesting, or even vigorously asserting that mathematics is either an art or science or even both. It is not only with those cases which categorically assert that mathematics *is* art that I shall be most concerned. I shall also be concerned with those views, which simply assert that some kind of *connection* between mathematics and art exists, and that this provides a rationale for learning mathematics.

For many writers it seems that to justify the study of mathematics, little more is required than to state or show a *connection* between the mathematics and art. In some extreme cases mathematics has been identified *as* art or art *as* mathematics. But however strong the connection, it is as though all that is necessary is to show some affiliation between them. I want to call this approach, *justification by association.* Not only have writers made connections of different *strengths* between mathematics and art, they have also made these connections between different *senses* of 'art'. So that the examples discussed will be of strong or weak connections between mathematics and art in a general or a specific sense. The first example is of a writer who makes a strong connection between mathematics and art in its general sense.

Justification by association with a general sense of 'art' - Fitch

We have already seen in Part 1 that Sir Joshua Fitch (1902), a nineteenth century inspector of training colleges, in his *Lectures on Teaching* wanted to distinguish between two different kinds of reasons, or rather 'objects', for learning arithmetic.[37] But Fitch went further than simply identifying these two objects. He linked them to the notions of *art* and *science*, and hence introduced his own categorisation of mathematics. Arithmetic he claimed is:

> ...both an Art and a Science:- an Art because it contemplates the doing of actual work, the attainment of definite and useful results; a Science because it investigates principles, because he who unearths the truths which underlie the rules of Arithmetic, is being exercised,[38] not merely in the attainment of a particular kind of truth about numbers, but in the processes by which truth of many other kinds is to be investigated and attained. (p.287)

This need to categorise mathematics, as I have remarked above, is understandable. Yet it is not always helpful, since writers vary in what they mean by 'art'. Fitch's view of science, with its emphasis on discovering laws and truths, is not unlike that of both Aristotle and Descartes, cited above. His view of art is less straightforward, though his reference to work, and what is practical, does suggest the use of the body to get something done, perhaps even to bring something into existence. We certainly regard *fine art* [39] as something of considerable value that has been brought into existence. So that to make the strong claim that mathematics is fine art would appear to provide us with a robust justification. But whatever value the fine arts have, it is not primarily usefulness. For Fitch, however, all that arithmetic brings into existence, if anything at all, are some 'definite and useful results'. So *his* use of 'art' is rather different from art in the specific sense of fine art. So what is a strong connection between arithmetic and art, loses force as a justification because the connection is made with art in a general sense.[40]

Indeed, despite the fact that Fitch believed that arithmetic is an art, and that this provided the first of his two 'objects' for teaching it, he also believed that arithmetic-as-an-art was of comparatively minor importance. This is because the aspect of arithmetic,

---

[37] Although Fitch focuses on arithmetic at this point, as his discussion develops he includes remarks about geometry, so we can safely assume that his remarks apply to mathematics in general.
[38] The use of 'exercise' here is of some interest. I shall discuss a certain ambiguity in the use of this word in Part 3.
[39] I shall be examining the notion of *fine arts* later. For the moment I am using it to highlight a difference in meaning from the general, descriptive sense in which Fitch uses it, and the more specific, high status notion of *the* arts.
[40] I call this a 'general' sense since to be a (fine) artist one has to learn certain practical arts which will be of use in producing works of art. In this respect the art of counting and measuring are analogous to the art

which for him characterised the subject as an art, is that which involves the carrying out of *routines*. He admitted that throughout history such routines have enjoyed considerable status, but in his time he believed that, unlike the 'arts' of reading and writing, they were of little value to the majority who learnt them:

> ... Counting - doing sums,- how often in life does this accomplishment come into exercise? Beyond the simplest additions and the power to check the items of a bill, the arithmetical knowledge required of any well-informed person in private life is very limited. For all practical purposes, whatever I may have learned at school of fractions, or proportions, or decimals, is unless I happen to be in business, far less frequently available to me in life than a knowledge, say, of the history of my own country, or of the elementary truths of physics.[41] (p291)

Nowadays, of course, more people are in a sense 'in business'. Yet widespread use is made of the electronic calculator, and the value of *standard* written methods for calculations has rightly been questioned. So Fitch is surely right to be seeking other justifications for mathematics, given his own view of the comparative uselessness of many mathematical routines. It is clear, then, that working with *his* notion of 'art', Fitch does not regard the fact that arithmetic is an art as bestowing any special value on mathematics. We might say that for him 'art' functions largely as a *description*. For many others, as I have suggested, to say that X is an art is at least to imply that X *is* of special value and that this special value is what justifies our learning it. So in this section, I want to look at the way in which the assimilation of mathematics to art, or at least the supposed affiliation between the two subjects, is supposed to provide an implicit justification for mathematics. Moreover, as we shall see, it is usually taken for granted that art is already valued, so that it is supposed that by connecting mathematics to art it is automatically justified.

Justification by association with a specific sense of 'art' - Kline

More recently, Morris Kline (1972) has viewed mathematics as art, but unlike Fitch, Kline views mathematics-as-art very highly:

---

of applying paint, drawing in perspective, being able to harmonise a melody, write in sonnet form etc. We could call this the 'instrumental' sense of art.

[41] This represents a direct attack on the scope of the utility view of mathematics particularly the view which links mathematics with commerce. We saw earlier in Part 1 how the application of mathematics in commercial contexts became overstated.

> For about a hundred years now mathematicians have come to recognise what was felt and asserted by the Greeks but had been lost sight of in the intervening centuries: *mathematics is an art* and mathematical work must satisfy aesthetic requirements.[42] (p.520)

With Kline we have our first statement of art which connects it to *aesthetics*, and although this connection is not uncontroversial it was missing from Fitch's account.[43] For Kline mere routines alone are not characterised as art. Something extra is implied. With the idea of aesthetics there is the suggestion of concepts such as beauty, elegance and others which connect art with the expression of feelings or emotion. Kline thus annexes mathematics to a specific sense of 'art'. If we provisionally allow a connection between aesthetics and beauty to be made then *some* support for Kline's reference to the Greeks [44] can certainly be found in Aristotle's remark that:

> ...the major Forms of the beautiful are order, symmetry and delimitation, and these are very much objects of the proofs of the mathematical sciences (Aristotle, 1998 edn p.400)

But all Aristotle is asserting here is that there are strong *connections* between mathematics and beauty. We cannot simply thereby assimilate mathematics into art, if only because not all cases of beauty are also cases of art, since beauty is found in the natural world.

What this shows is that if we use the association between art and mathematics as a justification we shall firstly need to be clear about the sense of 'art' that we are using. We shall have to be aware, too, of the history of the concept of art and the important distinctions between 'art' and what I have referred to as 'fine art'. But apart from clarifying the meaning of art used in these claims about mathematics, there is another task I shall undertake next. For some writers, the connections between mathematics and art arise from the dependency of art upon the existence and application of mathematics. Sometimes this has been put much more strongly when it has been asserted that art *is*

---

[42] My italics

[43] It should be pointed out that Fitch does recognise there are aesthetic qualities in mathematics. In particular he mentions beauty and harmony, but these are not mentioned until the very end of his account on mathematics, and as such they do not hold a central place in his account.

[44] Fitch, too, invokes the Greeks but only to argue that they did *not* view mathematics as art, or at least not in the sense of art that he insisted on using. (p.288)

mathematics. I shall trace this supposed connection between mathematics and art before returning to the claim that mathematics is a *fine* art.

Art as Mathematics – Plato and '*techne*'

In their introduction to Plato' s theory of art Hofstadter and Kuhns (1964) identify one dominant strand in his thought which associates mathematics with art:

> Art, conceived generally as *techne*, presupposes a knowing and a making: knowing the end to be aimed at and the best means for achieving the end. When a maker commands his art he can judge the excellence of his product according to his insight into proportion and measure. Fundamentally, then, the artist must, if he is to work well, know the nature of Measure. *Basic to any one art is the art of measure without which there can be no art at all.* For to know the proper length of a speech, the proper proportion of a painting, the proper distribution of functions in a society, the proper organisation of language in a poem, is to command the art of measurement. Measure for Plato embraces the principles of the good and the beautiful, and in our terms the principles of taste as well...Only the man who understands the fundamental principle of measure can judge which imitations are worthy, which debased.[45] (pp.3-4)

Here then we have one of the first statements of the connection between mathematics and art, where it is claimed that the existence of art, or at the least judgements pertaining to art, is dependent upon *some* mathematics. On this view, since art involves balance and proportion, and that these in turn rest on quantity and size, some competence in measurement is implicit in art. Thus aspects of art and mathematics are brought close together. But before we conclude that since art is of value, then mathematics by association must also be valued, we need to examine the nature of art referred to here, and the extent to which mathematics is connected to it.

In ancient philosophy what is translated as art is usually the Greek word *techne*, and was used to identify a much broader set of activities and objects than what we usually nowadays call art. In his editorial comments on Plato's *The Republic* (1955 edn), Desmond Lee writes of techne that:

> It may [thus] be said to cover any skilled activity with its rules of operation, the knowledge of which is acquired by training. But it is a very elusive word to translate, varying between art, craft, professional skill, and science according to the emphasis of the context (p.74)

---

[45] My italics

Art in this sense seems to be the way in which Fitch uses it. His view of arithmetic, as the art of carrying out routines of number for application, particularly in commerce, seems to be an ideal candidate for inclusion amongst techne. We may add that even though Fitch seems to have claimed that the Greeks did not view arithmetic as techne,[46] there seems to be no reason why they could not have done so. Instead, as Fitch and Kline both state in their different ways, the Greeks found a higher status for mathematics.

Clearly then, the amount and kind of mathematics which is essential for art as techne will vary. What is required need only involve a minimal amount of measuring skills for, say, the cobbler who fashioned shoes, and so we are back with something rather like a 'utility' justification for mathematics. Much, of course, will depend upon the art in question. The main point to note, is that we cannot simply read off an association of mathematics with art, from Plato's remarks, which can form a rationale for mathematics based on non-utility considerations.

One reason why we may want to resist the idea that art is closely related to mathematics is that nowadays art is often regarded as *expressing emotion* rather than displaying rationality. Yet the ancient concept of art was different. Art, for Aristotle (1953 edn), was 'nothing more or less than a productive quality exercised in combination with true reason' (p.175), which is in stark contrast with the more modern idea that art is often characterised by inspiration and emotion. Indeed, Plato, in his dialogue *Ion*, treated 'art' that depended upon inspiration with great suspicion. Thus Socrates says to Ion:

> ...this gift you have of speaking well [on Homer] is not an art; it is a power divine, impelling you like the power in the stone Euripides called the magnet...the lyric poets are not in their senses when they make these lovely lyric poems. (See Hamilton and Huntingdon, 1961 pp. 219-220)

For Plato, at least, music was scarcely an art for this reason. It only achieved such a status when it was realised that those aspects of music, such as harmony, have a mathematical and therefore rational basis.[47] Moreover, what counted as goodness in art,

---

[46] Fitch writes: 'Arithmetic, as taught in the schools of Athens or Alexandria; to the contemporaries of Socrates and Alcibiades; or later in the Middle Ages it shared with logic, geometry, grammar and rhetoric and music the distinction of forming one of the staple subjects of a liberal education, was taught in its principles, as a logical discipline; as something to be understood *rather than as a series of devices for working out problems*' (p.288)

[47] This identification is usually attributed to Pythagoras as it was pointed out in the Interlude.

for the ancients, was often that which was *functional*, so that one criterion for the arts was clear: a good artisan produced goods or services for a particular purpose. This is not true of the more contemporary concept of art. What we call art is usually non-functional, something to be enjoyed rather than used.[48] So we must be careful not to suppose that when the ancients connect art and mathematics, it is the same as the modern conception of fine art.

Art as mathematics - St. Augustine

A more distinctive account of the association of mathematics does arise, however, in the writings of St. Augustine, where the influence of Pythagoras is strong. As it was pointed out earlier,[49] little is known for certain of the precise doctrines attributed to Pythagoras. But one which has been passed down to us through Aristotle is the doctrine that 'everything is made of number', or more coherently that the rationale for the universe is a numerical one. Behind all appearance therefore lay a more ultimate reality. To apprehend this ultimate reality was to achieve wisdom. In the philosophy of Augustine the Pythagorean doctrine is connected explicitly to art. Augustine insists that the sensuous quality of nature reveals only part of its value:

> Wherever you turn, wisdom speaks to you...She does this through the forms of external things, leading you to see that all the material things which delight and woo you through the senses are characterised by number. (See Howie, 1969a p.250)

At first Augustine connects only the natural world with number:

> Turn your eyes to the heaven, the earth and the sea, to everything which shines brightly in or above them, which creeps, flies or swims. They all have forms, and this is because they have numbers; *take numbers away from them, and there is nothing left.*[50] (pp 250-1)

But then, in speaking about artefacts he echoes what was said about Plato's view above:

> Indeed, men who construct all physical shapes have numbers as the basis of their craft, and they make their products conform to them. (p.251)

---

[48] I say 'usually' because, for example, music might be composed for processional purposes or for dancing. In this respect we might want to say that a good composition was one which was especially well suited for these purposes.
[49] See my Interlude, above.
[50] My italics

But now we can say of Augustine, what might have been true for Plato but for which we had insufficient grounds for saying of the earlier writer, that the numbers which are basic to the craftsman, are not *subservient* to his or her art, but in a sense *constitute* it. In particular, Augustine wants to add that the pleasure which we receive from art, is at root, mathematical:

> ... if you ask what moves the limbs of the craftsman, the answer is "number," for even these move rhythmically. If the hands have no task to perform and the mind has no intention of making anything and if the movement of the limbs is solely for pleasure, then we call the activity dancing. *If you inquire what gives pleasure in dancing, number will reply. "Look, it is I."* Examine the beauty of a well-formed body; numbers are expressed there in spatial form. Study the beauty of physical movement; there you find number expressed in temporal form.[51] (p.251)

We have then in Augustine a clear statement that beauty, and thus aesthetic pleasure, is a function of mathematics. Indeed one contemporary critic has taken Augustine's remarks to be showing not simply that mathematics is a necessary condition of art, but that some *reduction* of art to mathematics has been implied. Thus Howie (1969b) comments:

> ...the standard of reference, by which Augustine judges beauty, is a numerical standard. Aesthetic pleasure derives from a perception of proportion and harmony, reflecting in natural things the thoughts of the Creator and in works of art the artist's grasp of these principles. *Both the creation and the enjoyment of works of art reduces itself, therefore, to a knowledge of number*, that is to an appreciation of the relationships of parts to wholes...[52] (p.118)

But, Howie does, however, quickly point out the inadequacy of Augustine's position:

> ...So far as it goes, Augustine's theory of aesthetics is satisfactory. But in emphasising the intellectual element in aesthetic pleasure, he explains only one aspect of an experience, which is more complex than he realised. The notion of number lying at the foundation of artistic creation needs to be supplemented by other considerations, of which Augustine does not take account. (p.118)

We may suppose that the 'other considerations', which Howie alludes to here, almost certainly include the feeling side of art which characterises many, if not all, more contemporary views of art. But despite the modern critic's view that Augustine's theory

---

[51] My italics

is inadequate, it is still important to point out that his was a view once held, and that it does seem to have had some influence on those who connect art and mathematics.[53] Indeed, at least one writer, Morris Kline, insists that emotion is *not* central to the concept of art, and he seeks other properties of mathematics which resemble those of art. But before we consider in detail what he has to say, we shall consider the views of other writers who, within the last hundred or so years, explicitly connect mathematics with the fine arts.

---

[52] My italics

[53] We shall also see that what appears to be Augustine's influence is present in one important theory of aesthetics in mathematics. (see my later discussion of Charlton's work)

Chapter 5 – Mathematics as a Fine *Art*

I noted, in Part 1, when I was discussing the work by Griffiths and Howson (see page 36), that in maintaining a distinction between the practical and aesthetic aspects of mathematics, they allowed the notion of the aesthetic to have a large scope. Nevertheless, whatever else these writers subsumed under the concept of the aesthetic, in connection with mathematics, the *fine arts* was not one of them.

Yet several other writers have explicitly claimed that mathematics, at the very least, *resembles* the established forms in which fine art is most typically cast. Some writers go even further and claim that mathematics *is* a fine art. In speaking of *fine* art, such writers are clearly not simply referring to the 'art' of mathematical routines mentioned earlier. Nor are these writers, simply admitting that mathematics is an ingredient in art, or even that artists require competence in various aspects of mathematics. They make a strong connection between the high status fine arts and mathematics, so that 'added value' is attached to the latter by association. Thus conceived, mathematics is not simply a tool, but a highly valued end in itself. Clearly, if such claims could be shown to hold, then equating mathematics and fine art would provide an important rationale for mathematics.

Three characteristics are often given of fine art, namely, representation, expression and form. The most plausible of these in relation to mathematics is form, and that is the one which will ultimately be argued for here.[54] It is not clear how mathematics could express emotion. However it may be worth suggesting, in passing, a possibility of making a connection between the idea of representational art and mathematics. The applied mathematician makes a mathematical model of reality which represents the world in some way. The mathematics does not, however, have sensuous properties which resemble the world, rather the mathematics mirrors certain formal relationships of quantities. It is just possible that we could argue that mathematics was a work of art in this respect, though I know of no one who has attempted to make such a connection. Moreover, it would be difficult to see how this would provide a justification for learning mathematics which did not invoke the idea of a tool and hence 'usefulness'. In

[54] This will not be revealed, however, until I develop my account away from the fine arts to the distinguishable area of aesthetics.

what follows it should be clear that it is the formal properties which are most prominent in the discussion.

The Moritz anthology

Robert Edouard Moritz has collected an anthology of extracts, by mathematicians and writers on mathematics. Thirty or so of these form a chapter entitled 'Mathematics as a Fine Art' (Moritz 1914). Moritz gives no commentary on the chapter, many of the contributions to which were written in the nineteenth century, and hence the reason for the inclusion of any particular piece is not made clear. Whether one author or another, of a particular fragment, is successful in showing that a strong connection between mathematics and the fine arts exists, or whether indeed mathematics ought to be considered as fine art, is not endorsed by the editor. Yet the chapter, left uncriticised as it is, does give the impression that the contributors have justified the study of mathematics by association with the fine arts. So it is worth examining carefully the selections from the chapter, in order to see, for example, whether there are important and useful common features amongst them, and whether such features provide the basis of a convincing account.

At the outset, it is interesting to note that Moritz, or indeed any other writer, should explicitly connect mathematics with the *fine* arts. The expression 'fine arts' was introduced by Charles Batteux in the middle of the eighteenth century in order to refine the more general term 'art', the meaning of which is very broad as we have seen. According to Hanfling (1992) the ancients did not distinguish between art in general and a specific class of fine art. As he points out:

> It is usually held that the decisive and most influential statement of the 'modern system' was that of Charles Batteux, who, in his treatise *Les beaux arts réduits a un même principe* ('the fine arts reduced to a single principle') of 1746, separated the 'fine arts' from the 'mechanical arts', and listed the former as consisting of music, poetry, painting, sculpture and the dance. Batteux tried to show that the principle common to the fine arts was the 'imitation of beautiful nature'. But his main influence lay, not in the particular principle that he put forward, but in his clear separation of the fine arts - the arts, as we might say - from other activities, according to some principle or definition. What distinguished the 'modern system' was not that it replaced some previously existing system, but the very idea of treating the arts in a systematic way. (p.7)

According to this original definition, then, mathematics as it stands is certainly *not* a fine art. So either the term 'fine art' must be extended to include things which were not originally included, or it must be shown in some *special way* that mathematics is, after all, one of the objects on the original list of fine arts, or at least that it closely resembles one or other of such objects. If the concept 'fine art' needs to be modified in order to include mathematics, we would need to ask whether such a modification is justified or whether it destroys the purpose of setting out such a small exclusive list.[55] We should bear in mind, however, that what constitutes the fine arts can change from time to time whilst still preserving the spirit of Batteux's list. In more recent times it might include, for example, cinema and photography.[56] However, the writers in Moritz's anthology try to persuade us that mathematics shares common features with painting, sculpture, literature (in particular poetry) or music, thereby suggesting that mathematics is associated with items on Batteux's original list.

Personal characteristics: Fine artists and mathematicians

Several of the contributors to the anthology focus on *mathematicians* rather than mathematics. So that in drawing a resemblance between the mathematician and the artist, they imply that mathematics should be viewed as a fine art. Thus Moebius, giving us very little of substance, writes:

> It is with mathematics not otherwise than it is with music, painting or poetry. Anyone can become a lawyer, doctor or chemist, and as such may succeed well, provided he is clever and industrious, but not every one can become a painter, or a musician, or a mathematician: general cleverness and industry alone count here for nothing. (Moritz, 1914 p.184)

But all that follows from this observation is that both the mathematician and the artist are set apart from others by dint of their special qualities. We are given no further information about the *nature* of such qualities. Bocher, on the other hand, is more explicit about the kinds of common qualities that are shared by fine artists and mathematicians:

---

[55] It should become clear later, that I do not believe that the concept of fine arts can be extended to include mathematics without the risk of losing something of special value in the arts.
[56] For a cautionary remark on photographs and fine art from a writer who energetically argues for a strong connection between mathematics and the fine arts, see Kline (1972) p521

> I like to look at mathematics almost more as an art than as a science; for the activity of the mathematician, constantly *creating* as he is, guided though not controlled by the external world of the senses, bears a resemblance, not fanciful I believe but real, to the activity of an artist, of a painter let us say.[57] (Moritz, 1914 p.182)

For Bocher, as for many mathematicians, mathematics is not simply discovered as is sometimes ordinarily supposed. It is, to some extent at least, *created*, and it is thus creativity which is used here to connect artist and mathematician. Others writers in Moritz's anthology are only slightly more forthcoming in saying precisely *what* it is that is being created by the mathematician. For example, Lampe writes:

> In the projection of new theories the mathematician needs as bold and creative a phantasy as the productive artist, and in the execution of the details of a composition the artist too must calculate dispassionately the means which are necessary for the successful consummation of the parts. Common to both is the creation, *the generation, of forms out of the mind.*[58] (Moritz, 1914 p.185)

But even if we can make sense of the mathematician creating theories and forms 'out of the mind',[59] the idea of a *creative* mathematician, does seem to have what Elliott (1971) has called 'ironic undertones'. It is an ambiguous notion. This is because there are, according to Elliott, two quite distinct versions of the concept of creativity. One of these versions, the so-called 'traditional concept' essentially involves the *making* of something, and hence it is particularly appropriate to the artist, but is misapplied if used in certain other contexts.[60] He writes:

> If we call someone a 'creative' historian it is virtually impossible, no matter how we load the context, to avoid the suggestion that he makes up his stories instead of deriving them from the historical evidence...'Creative biologist' suggests a breeder of new germs; 'creative anatomist' a Dr. Frankenstein; 'creative chemist' an alchemist. (Elliott, 1971 p.140)

The same kind of irony or ambiguity, we might add, can be attached to the idea of the 'creative mathematician', rather as it is to the idea of a 'creative accountant'. It suggests the idea that the truth has somehow been displaced. Yet one writer in the anthology,

---

[57] My italics
[58] My italics
[59] The underlying idea here of constructing 'forms out of the mind' is clearly related to my earlier point that the traditional definitions of art and science can in principle be merged.
[60] See for example the definition of art in Aristotle (1953 edn) p175.

Thomas Hill, seems to be so determined to connect art and mathematics that he is prepared to accept this:

> The Mathematics are usually considered as being the very antipodes of Poesy. Yet Mathesis and Poesy are of the closest kindred, for they are both works of the imagination. Poesy is a creation, a making, a fiction; and the Mathematics have been called, by an admirer of them, the sublimest[61] and most stupendous of *fictions* [62]... (Moritz, 1914 p.189)

However, Elliott does point out that not all applications to science, of the traditional concept of creative, are ambiguous. There is it at least one class of notable exceptions:

> Although nowadays most educated people do not think of the practitioners of normal science as creative, they do regard *revolutionary* scientists - men like Newton, Darwin, Einstein and Freud - as creative, and, what is more, believe them to have been creative to an exceptionally high degree. (Elliott, 1971 p.144)

This is so, according to Elliott, because great artists do not simply make *things*. We regard them more as having made a *world.* This accomplishment he believes can be attributed to great scientists too, those who:

> ...have quite radically re-structured *our* world, which is *the* world as we conceive - and even perceive - it. (Elliott, 1971 p.144)

Alongside the revolutionary scientists whom Elliott cites, it would seem reasonable to include certain mathematicians, who in the same way we could intelligibly regard as being creative. Amongst these we could include, for example, those mathematicians who have shown us that new and different geometries are possible, and that Euclidean geometry does not finally and conclusively describe the space of the world in which we live, as was once unshakeably believed. This way of viewing the matter begins to remove some of the possible ambiguity in what has been said so far, when it has been suggested that mathematicians, like artists, are characterised by their creativity. Thus Pringsheim, in his contribution to the anthology, is more explicit in what the mathematician creates when he writes:

---

[61] The notion of the sublime in aesthetics is discussed briefly below with the concept of beauty.
[62] My italics

> The true mathematician is always a good deal of an artist, an architect, yes, of a poet. Beyond the real world, though perceptibly connected with it, mathematicians have intellectually *created an ideal world*, which they attempt to develop into the most perfect of all worlds, and which is being explored in every direction. None has the faintest conception of this world, except he who knows it.[63] (Moritz, 1914 p.184)

In addition to creativity, Bocher, also cites *imagination* as a quality shared by fine artist and mathematician:

> Just as no one can become a good painter without a certain amount of skill, so no one can become a mathematician without the power to reason accurately up to a certain point. Yet these qualities, fundamental though they are, do not make a painter or mathematician worthy of the name, nor indeed are they the most important factors in the case. Other qualities of a far more subtle sort, chief among which in both cases is *imagination*, go to the making of a good artist or good mathematician.[64] (Moritz, 1914 p.182)

Hill, also stresses the importance of imagination:

> Mathesis and Poetry are...the utterance of the same power of *imagination*, only that in the one case it is addressed to the head, and in the other, to the heart.[65] (Moritz, 1914 p.189)

But to invoke the *imagination* is perfectly apt in mathematical contexts. Indeed, the concept of imagination is closely linked with what Elliott refers to as the 'new' concept of creative, one which does not entail the making of something. This new concept of creative has two main versions:

> According to the first of these, a problematic situation is defined as one for which no adequate response is available in terms of existing knowledge, methods and techniques, and creativity is taken to be the capacity to resolve situations of this kind. The second version identifies creativity with getting novel ideas and making something of them. By 'making something of' an idea is meant either solving some existing problem by means of it, or putting it to some other acceptable purpose, or just making it available to others who actually do or might well find some such employment for it. (Elliott, 1971 p.146)

Elliott adds:

---

[63] My italics
[64] My italics
[65] My italics

> ... under the new concept to proceed imaginatively is *ipso facto* to be creative. All creativity is creative (i.e. imaginative) thinking. (Elliott, 1971 p.147)

So that now we may apply the new concept quite readily to science:

> Hitherto we might have said of a scientist or scholar that he had performed original work in which he had used imagination to solve his problems, but we would not have called him creative. Under the new concept he qualifies for this latter title. (Elliott, 1971 p.147)

On the other hand this new concept does not sit well with art. He writes:

> ... the acceptability of the new concept to the artist and lover of art depends on whether artistic activity can be adequately conceived in terms of problem-solving or getting novel ideas, i.e. in terms derived from the analysis of scientific enquiry. *The danger is that by assimilating art to science we shall misconstrue the nature of art...*[66] If what counts as a problem has no reference to what the artist experienced as a problem, any and every element in the work can be regarded as a solution of a problem...though it has analogies with scientific enquiry, artistic creation also proceeds differently, within a different encompassing form of life. *The new concept may not be so well-suited to art as the traditional concept was.*[67] If we assume or presuppose that it *is* well-suited to art, this may lead us to analyse artistic creation in a manner which distorts our understanding of it. (Elliott, 1971 p.148)

The notion of innovation has certainly become important in art. Hanfling (1992) cites Dubuffet the painter who wrote: 'The essence of art is novelty. Likewise should views on art be novel. The only system favourable to art is permanent revolution'. (p.4) So with this in mind perhaps we cannot rule out mathematics as a fine art if it involves the development of novel ideas, as it surely does in some of its aspects. Yet Elliot continues:

> The view that artistic creativity is a matter of getting and making use of novel ideas is open to a rather different objection. This view identifies artistic creativity with inventing novel techniques or new methods of composition or exploiting the medium in new ways - in a word with changing the language of the relevant art. (Elliott, 1971 p.149)

Referring to an evaluation of the work of T.S. Eliot and that of Yeats he remarks that:

---

[66] My italics
[67] My italics

> Eliot has contributed more than Yeats to the language of twentieth century poetry, yet many critics would want to say that Yeats is the greater and more creative poet. Such cases, where criteria of creativeness conflict, are incomprehensible if we think in terms of the new concept alone. (Elliott, 1971 p.149)

From this discussion it does seem that an artist can be creative in both senses, for presumably T. S. Eliot did 'make' poetry as well as finding new form. The mathematician, too, can be creative in both senses according to another contributor to the anthology:

> Who has studied the works of such men as Euler, Lagrange, Cauchy, Riemann, Sophus Lie, and Weierstrass, can doubt that a great mathematician is a great artist? The faculties possessed by such men, varying greatly in kind and degree with the individual, are analogous with those requisite for constructive art. Not every mathematician possesses in a specially high degree that critical faculty which finds its employment in the perfection of form, in conformity with the ideal of logical completeness; but every great mathematician possesses the rarer faculty of constructive imagination. (Moritz, 1914 p.184)

Elliott's two concepts of creative are echoed here in Hobson's remarks. Clearly, he likens only the leading mathematicians to the great artists and those have a distinctive kind of imaginative faculty. Others writers who have drawn attention to the importance of the imagination, like Sylvester, have a more simple idea in mind:

> Surely the claim of mathematics to take a place among the liberal arts must now be admitted as fully made good. Whether we look at the advances made in modern geometry, in modern integral calculus, or in modern algebra, in each of these three a free handling of the material employed is now possible, and an almost unlimited scope is left to the regulated play of *fancy*.[68] (Moritz, 1914 p.183)

The upshot of this is that the concept of creativity only provides a link between the mathematician and artist if we understand creativity in certain special ways. The version of the concept as usually applied to artists, applies to a certain few outstanding mathematicians. In its transformed version it applies to all mathematicians but not to all artists. Moreover, the two versions of the concept accentuate, rather than remove, the difference between the mathematician and artist. But why should the mathematician want to be regarded as creative? Why should he or she want to be allied to the artist anyway? I have assumed from the start that to assimilate mathematics into the fine arts

---

[68] My italics

would add value to mathematics. Contrary to this Elliott, now adopting a rather ironic tone himself, writes:

> But the scientific discoverer and the *savant* have never been regarded as having a status inferior to that of the artist. They have not been called 'creative', but their understanding of nature, of the scriptures and of the classical past have usually been valued much more highly than the artist's creativeness. It has never been forgotten that the worlds created by the artists are unreal, and there has been a persistent tendency to place the creation and enjoyment of such objects outside the really serious business of life. (Elliott, 1971 p.144)

Although Elliott is comparing scientist and artist, and we are comparing mathematician and artist, his point may be extended to include mathematics. The mathematician may not claim to have precisely the same kind of understanding that Elliott attributes to the scientist or scholar. But he or she can quite rightly claim to have a profound understanding of other analogous objects, such as number properties and the relationships between them.

So far we have seen that the indirect way of connecting mathematics to the fine arts by comparing artist with mathematicians, as some of the contributors to Moritz's anthology have done, is not without its difficulties. Suppose that we compare particular kinds of fine art and mathematics, rather than their practitioners, perhaps this will provide a clearer more convincing association between the two enterprises. Painting and music have both either been directly referred to, or have at least been alluded to in some of our examples so far. In several other extracts from the anthology attempts are also made to reveal a strong connection between mathematics and music. Let us see whether or not these are convincing.

Certainly nothing is forthcoming in the assertion by Novalis who simply writes the following: 'Music has much resemblance to algebra' (Moritz, 1914 p.190). Helmholtz's remarks, though more extended, are also not illuminating:

> Mathematics and *music*, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all activities of our mind, and which leads us to surmise that the manifestations of the artist's genius are but the unconscious expressions of a mysteriously acting rationality.[69]
>
> (Moritz, 1914 p.191)

The same goes for Sylvester, who gives us no clear idea of the link, either. Indeed he gives us quite a strong reason for showing that art remains the domain of the senses, and mathematics the domain of reason:

> May not Music be described as the Mathematic of sense, Mathematic as Music of the reason? the soul of each the same! Thus the musician *feels* Mathematic, the mathematician *thinks* Music, - Music the dream, mathematics the working life - each to receive its consummation from the other when the human intelligence, elevated to its perfect type, shall shine forth glorified in some future Mozart-Dirichlet or Beethoven-Gauss - a union already not indistinctly foreshadowed in the genius and labours of a Helmholtz! (Moritz, 1914 p.191)

It rests with Cajori to give us anything substantial by way of a comparison. His point is the Pythagorean one that number relationships underlie music. Yet the most that this point can show is that mathematics is really a part of music, as we have seen in Augustine's theory. It can hardly show that mathematics *itself* is fine art:

> ...he [Pythagoras] endeavoured to discover some principle of homogeneity in the universe...he observed that musical strings of equal lengths stretched by weights having the proportion of 1/2, 2/3, 3/4, produced intervals which were an octave, a fifth and a fourth. Harmony, therefore, depends on musical proportion; it is nothing but a mysterious numerical relation. Where harmony is, there are numbers. Hence the order and beauty of the universe have their origin in numbers. 1130 (Moritz, 1914 p.190)

For Poincaré, too, it seems that mathematics has '...delights *analogous* to those that painting and *music* give'[70] (Moritz, 1914 p.181). Yet we are given no more detail of the nature of the analogy between mathematics and music. But whatever the 'delights' were, he was clear that they were only to be enjoyed by the elite few, or what he called 'the adepts'. This restricted accessibility to certain features of mathematics, which is especially associated with Poincaré, is something which I shall take up again later. But

---

[69] My italics

we should note here that others in the same anthology seem to hold a different opinion from Poincaré, provided that we suppose that the 'delights' of mathematics, which Poincaré had in mind, include what Young refers to as its 'beauties':

> The beauties of mathematics - of simplicity, of symmetry, of completeness - can and should be exemplified even to young children.[71] (Moritz, 1914 pp. 184-185)

Not all of the writers in the anthology are in agreement on *which* of the fine arts mathematics most resembles. Several writers, as we have seen, have linked painting and music with mathematics. But these particular links are rejected by Bertrand Russell in favour of his own:

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of *sculpture*, without appeal to any part of our weaker nature, *without the gorgeous trappings of painting or music*, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.[72] (Moritz, 1914 p.182)

In trying to annexe mathematics to the fine arts, difficulties emerge in certain of the ways in which writers try and draw out the supposed connection between the two enterprises. The linking of mathematics and art by the apparent similarities in the capacities of its practitioners, we have seen is by no means convincing. The predominant capacity invoked is that of creativity and there is not a satisfactory match between the versions of this concept most appropriate to artist and mathematician except in the rarest of cases. Nor have any clear analogies been made between mathematics and particular fine art forms. Moreover, even if mathematics were considered to be fine art, it is not altogether clear whether or not it is sufficiently *accessible* enough for such a consideration to be embodied in a rationale for learning the subject.

Aesthetic Properties: Beauty and the Sublime

It was necessary to distinguish, earlier, between the very general concept of *an* art, which included the mastering of certain routines, and a more recent view which invoked

---

[70] My italics
[71] Recall that contemporary educationists like Leibeck (1984) believe that intrinsic features like aesthetic satisfaction, rather than utility, provide the more pressing reasons for children engaging in mathematics.
[72] My italics

what Morris Kline called 'aesthetic requirements'. Typical amongst such requirements is *beauty*, which has often been seen as an essential property of art. Indeed, we have just seen that Bertrand Russell wanted to make a strong link with *sculpture* by drawing our attention to what he believed was the 'supreme', 'cold and austere' beauty of mathematics. There are, however, difficulties in trying to suggest that mathematics is a fine art by appealing to certain of its aesthetic properties. To reiterate a point made earlier, beauty cannot be a sufficient condition of fine art since there exist objects of *natural beauty*, like flowers and sunsets, so the presence of beauty alone in an object does not guarantee its inclusion amongst the fine arts. Not all of the writers in Moritz's anthology make a connection between beauty in mathematics and the fine arts.[73] Indeed Kummer, in the following remarks *distinguishes* mathematics from art:

> A peculiar *beauty* reigns in the realm of mathematics, *a beauty* which resembles not so much the *beauty* of art as the *beauty* of nature and which affects the reflective mind, which has acquired an appreciation of it, very much like the latter.[74] (Moritz, 1914 p.185)

Sylvester, too, in applauding the beauty of mathematics does not go as far as connecting the subject to fine art:

> ... the contemplation of *divine beauty* and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which it is concerned, these, and such like, are the surest grounds of the title of mathematics to human regard...[75] (Moritz, 1914 p.181)

But other references to beauty in mathematics are less cautious. Although Russell, in likening mathematics to sculpture, makes one of the strongest links between mathematics and the fine arts, in Moritz's anthology, Young, with his use of 'works' suggests a similar kind of link:

---

[73] It is important to note that one reason I am carrying out this analysis of Moritz's extracts is to see whether there are grounds for including mathematics amongst the fine arts. But not all of his extracts insist that that mathematics should be categorised thus.

[74] My italics

[75] My italics

> Mathematics has *beauties* of its own - a symmetry and proportion in its results, a lack of superfluity, an exact adaptation of means to ends, which is exceedingly remarkable and to be found elsewhere only in the works of the greatest *beauty.*[76] (Moritz, 1914 p.184)

But beauty does not even appear to be a necessary condition of fine art. Some fine art displays features which writers have contrasted with beauty, namely, those features which are *sublime.* For certain Eighteenth century writers such as Edmund Burke it was the *vastness* of sublime objects which distinguished them from beautiful objects which were generally regarded as being essentially more confined. Whether or not this is a satisfactory distinguishing feature to make, it has, nonetheless, been of some historical importance because, as Hanfling (1992) remarks:

> The assumption that beauty is a necessary condition of art – or rather, of good, successful art – could now be challenged, and various other qualities put forward by which to judge and appreciate a work of art. (p.44)

Kant, too, gave some consideration to the notion of the sublime and even linked some experiences of the sublime to aspects of mathematics. In distinguishing one of the two kinds of sublimity which were identified by Kant, Warnock (1976) writes:

> There is one sublimity, the feeling of which arises from *the contemplation of vast numbers…Infinitely large numbers give us a sense of the sublime* because in contemplating them the imagination calls to our mind the powers of reason, in contrast with its own feebleness. By reason we are not incapable of calculating with large numbers, but we can never make these calculations concrete by actually envisaging the numbers… [77] (p.58)

So, mathematics may be a source of both beauty and the sublime. Indeed, in Moritz's anthology both of these features are acknowledged, in what Boltzmann has to say, although his concept of the sublime is fairly muted:

---

[76] My italics

[77] My italics

> *Beauty*, I hear you ask, do not Graces flee where integrals stretch forth their necks? Can anything be *beautiful*, where the author has no time for the slightest external embellishment?... Yet it is this very simplicity, the indispensableness of each word, each letter, each little dash, that among all artists raises the mathematician nearest to the World-creator; it establishes a *sublimity* which is equalled in no other art, - something like it exists at most in symphonic music.[78] (Moritz, 1914 p.186)

There is of course no contradiction in saying that mathematics is both beautiful and sublime. Nature, too, has beauty and, in its dramatic landscapes, for example, it also has sublime aspects. What is important, however, is the significance of any such duality in the aesthetic properties of mathematics for its justification. The fine arts we have supposed are objects of pleasure, this is at least *one* reason why they are valued.[79] But it is not clear whether or not sublime properties are best regarded as objects of pleasure. As Warnock (1976) points out:

> Kant argues that it is more proper to describe the sense of the sublime as producing not pleasure so much as awe or respect. For what we respect is the idea of reason itself. What we stand in awe of is the fact that we are such as to be able to frame such ideas. (p.58)

There is another difficulty attached to the identification of beauty. We have been exploring this property in order to see whether it can lead us to an important non-utilitarian justification for learning mathematics. But in Young's extract, quoted above, one of the ways in which mathematics has beauty is in its 'exact adaptation of means to ends'. Curiously, this suggests that beauty might be connected with utility, something that we might suppose is diametrically opposed to the fine arts.[80] This is not a new idea. Plato in his dialogue, *Greater Hippias*, allows Socrates to equate usefulness and beauty:

---

[78] My italics

[79] I shall consider other values of the arts when I consider the views of John White.

[80] I realise that in this particular instance the ends involved may not imply *practical utility*. We have already seen that it is possible to have usefulness *within* mathematics. But the next ascriptions of beauty *are* premised on practical ends.

> ... we say that the whole body is beautifully made, sometimes for running, sometimes for wrestling, and we speak in the same way of all animals. A beautiful horse, or cock, or quail, and all utensils, and means of transport both on land and on sea, merchant vessels and ships of war, and all instruments of music and of the arts generally, and, if you like, practices and laws - we apply the word 'beautiful' to practically all these in the same manner. In each case we take as our criterion the natural constitution or the workmanship or the form of enactment, and whatever is useful we call beautiful, and beautiful in that respect in which it is useful and for the purpose for which and at the time at which it is useful, and we call ugly that which is useless in all these respects.
>
> (See Hamilton and Huntingdon, 1961 pp. 1548-1549)

The view that 'whatever is useful we call beautiful' has even been applied to mathematics recently. Davis and Hersh (1980*)* in order to illustrate what they call 'Mathematical Maoism', quote an extract from a dialogue on the beauty of mathematics from a discussion at the Shanghai Hua-Tung University, between an American professor of mathematics, J.J. Kohn, and a delegate during a conference of the People's Republic of China:

> *Kohn*: Should you not present the beauty of mathematics? Couldn't it inspire students? Is there room for the beauty of science?
>
> *Answer:* The first demand is production.
>
> *Kohn*: That is no answer.
>
> *Answer:* Geometry was developed for practice. The evolution of geometry could not satisfy science and technology; in the seventeenth century, Descartes discovered analytical geometry. He analysed pistons and lathes and also the principles of analytical geometry. Newton's work came out of the development of industry. Newton said, "The basis of any theory is social practice." There is no theory of beauty that people agree on. Some people think one thing is beautiful, some another. Socialist construction is a beautiful thing and stimulates people here. Before the Cultural Revolution some of us believed in the beauty of mathematics but failed to solve practical problems; now we deal with water and gas pipes, cables, and rolling mills. We do it for the country and the workers appreciate it. It is a beautiful feeling. (p.88)

Here the notion of beautiful has been extended, until it is simply a term of commendation. So that geometry may be seen as beautiful because it was originated for practical purposes. Yet objects of fine art are typically, if not exclusively, objects of contemplation, rather than things which serve a practical purpose. Moreover a 'beautiful feeling' is of course a different thing from a feeling of beauty. The former can amongst other things simply mean a feeling of relief. It is an example of the use of beautiful in its most general sense in which almost anything can be beautiful.

Despite these difficulties in the use of 'beauty' in mathematics, many writers persisted in claiming that mathematical beauty exists. All I want to suggest at this point is that even if we admit that mathematics has beauty, and that such beauty is not consequent upon its utility, this would not be sufficient to show that mathematics is therefore a fine art. It might however be tantamount to accepting that mathematics has certain *aesthetic* aspects. But in order to show that the presence of aesthetic properties does not imply the presence of fine art, it is necessary to show that such properties are not coexistent with art. This I shall do after I have finally determined whether or not mathematics may be regarded as fine art.

A more recent account

As I have already pointed out, the extracts from Moritz's anthology, were not written recently. However, attempts to connect mathematics and fine art still persist. Moreover, as we shall see, the compelling features which persuaded mathematicians of the last century that mathematics was a fine art are those which recent writers have invoked. Thus Morris Kline (1972), leans heavily on the capacities and personal qualities of mathematicians in his account. But since Kline's account is much more elaborate than any of the fragments in Moritz's anthology it is worth examining. Not only does he supplement the creative aspect of mathematics with certain other properties to bring mathematics closer to art, he also makes a bold attempt to change our view of some of the fine arts in order to bring them closer to mathematics.

Kline begins by asserting that all branches of mathematics are characterised by a particular *method,* one which uses a set of *axioms* as its starting point and upon which all the results are built, using logical steps. For centuries the axioms of geometry arose from our experiences of the world and were believed to be validated in the same way. In particular, geometry was thought to assert *truths* about the space which surrounds us. Central to Kline's argument is the fact that this axiomatic method took on a special significance when mathematicians developed consistent systems the axioms of which did not derive their truth from the physical world. So that mathematics, then, was no longer seen as the discovery of a system of truths. It became a logical system which had, as it were, broken away from its roots. One important consequence of this, according to

Kline, is that mathematics ought to be regarded more as a fine art than as a descriptive science.

Kline (1972) points out, however, that the logical method of mathematics, alone, does not sufficiently account for the way in which its results are originated and developed. In particular logic cannot determine *what* theorems to prove, nor can it determine *how* to prove such theorems. For the most part psychological processes of imagination, insight, creativity and inspiration must enter into the picture. As he remarks:

> Most people could look at a quadrilateral indefinitely without becoming aware that if the midpoints of the four sides are joined, the figure formed is a parallelogram. Such knowledge is not the product of logic but of a sudden flash of insight. (p.511)

It is partly this dependence upon inspiration that leads Kline to believe that the mathematician is indistinguishable from the composer.[81] '...essentially both mathematician and composer', he writes, 'are moved by a divine afflatus that enables them to 'see' and 'know' the final edifice before one stone is laid.' But the breaking away of the axioms from their roots in experience, and the invention of others which do not necessarily arise from such experience, has intensified this belief that the mathematician is like an artist, a creator rather than a discover. As Kline explains:

> The creation of non-Euclidean geometry...released the mathematician from the bondage of producing truths and set him free to adopt axioms and to investigate ideas that may have no apparent usefulness in mastering or understanding the physical world. And so the mathematician is compelled to ask himself what guides his choice of subject matter and what motivated his activity. (p.520)

Kline, as I have said, believed that mathematicians of modern times are motivated in a similar way to that of the Greeks, namely, by what he has called the satisfying of 'aesthetic requirements'. But as we saw earlier, art was of a rational nature for the Greeks. To reiterate, it was for Aristotle (1955 p175) 'nothing more or less than a productive quality exercised in combination with true reason' and hence in apparent contrast with the ideals of the fine arts which has been developed over the last two centuries. To suggest, therefore, that Greek ideals underpin the mathematicians' construction of axiomatic systems may have little or no bearing on the question of

---

[81] This reference to divine inspiration, recall, is in stark contrast to the way that Plato regarded the source of art. (See Ion pp 219-220 in Hamilton and Huntingdon, 1961)

whether mathematics is a *fine* art. So if Kline does want to insist that mathematics is art, in the modern sense of fine art, then it is not clear how he can avoid addressing the *feeling* element, which is so important to much if not all of art. It is important to note that Kline does acknowledge this point and does not suppose that 'aesthetics properties' and 'art' are synonymous. He realises that the usual assumption that art has an emotional element which is missing in mathematics is a potential stumbling block in trying to bring mathematics and art together. Kline attempts to overcome this difficulty by identifying an emotional element in mathematics, on the one hand, and by diminishing the importance of emotion in the fine arts, on the other. With regard to the emotion in mathematics he writes:

> No doubt many people feel that the inclusion of mathematics among the arts is unwarranted. The strongest objection is that mathematics has no emotional import. Of course this argument discounts the *feeling of dislike and revulsion* which mathematics induces in some people. This argument also undervalues the *delight* experienced by creators of mathematics when they succeed in formulating their ideas and in erecting ingenious and masterful proofs. Even the student of elementary mathematics is *pleased* by his success in proving stereotyped exercises and by his ability to see light, meaning, and order where formerly there was obscurity and confusion.[82] (p.521)

It must be said that this is a very weak attempt at showing that mathematics has an emotional element. The negative feelings which Kline refers to are not intrinsic to mathematics. The positive feelings, on the other hand, which are experienced in the production of mathematics, do have something in common with those experienced by artists. But whereas the artist can additionally, intentionally, express emotion in his or her work, this is not true of the mathematician. Moreover, to say that the student is 'pleased' with their efforts may have very little to do with the mathematics; the same pleasure could arise from any demanding activity.

So Kline's case has not yet been made, and this much he concedes. Perhaps his attempt to remove the kinship of emotion and art will fare better. His first move in this direction is to admit that 'a person is logically able to insist that the primary function of art is to arouse emotions and stir feelings'. But he points out that this function cannot be *sufficient* to pick out works of art since 'a dramatic photograph' might move us more

---

[82] My italics

than certain great works of art.[83] It appears to Kline that the arousal of emotions is not *necessary* to the arts either. He presses this point by giving precise examples of works of fine art which he believes do not satisfy this requirement:

> The still-life paintings of Picasso, impressionistic studies, such as Monet's, of atmospheric and light effects, the work of Seurat and Cézanne, and the 'arrangements' of the Cubists would also fail to satisfy the requirement. In fact, the pure art of modern times puts emphasis on the theoretical and formal side of painting, on the use of line and form, and on technical problems. Such work appeals much more to the intellect than to the emotions. (p.521)

But whilst it might be true that these works do appeal to the intellect to *some* extent, to say that they appeal *more* to the intellect than the emotions seems to be loading the die too much in favour of Kline's own position on mathematics. For Kline, moreover, the emotional aspect of art is largely attributable to the work of the Renaissance. This is clearly an exaggeration. Even if it were true of paintings, it is false of music. No one would surely deny that intense emotion was present in much of nineteenth century music. One only needs to think of the works of Chopin, Brahms or Tchaikovsky. It is true that much of contemporary music has been called 'cerebral', but our concept of music has not changed dramatically. There is still an overriding demand for works of emotional content in our concert programmes, and as we try to come to grips with a new music concept - even that which consist solely of silence - there are insufficient grounds for concluding that emotion is minimal in the arts even if it is missing in some of them.

However, Kline does more than stress the negative point that since emotion and art are only contingently connected, mathematics cannot be ruled out as a fine art in this respect. He sets out a positive reason why mathematics is art, namely, that it is 'an outlet for the creative instinct of man. But as we have seen, the concept of creative has different versions and these versions tend to highlight the difference between artist and mathematician rather than draw them together. Kline adds no more to the concept of creative to convince us otherwise. Indeed he makes the important point that creativity

---

[83] I remarked earlier that the fine arts may be extended to include such media as photography. It is interesting to see that Kline seems to find it less plausible to regard a 'dramatic photograph' as art than mathematics.

alone is not sufficient, and that the mathematician must produce work with 'design, harmony and beauty'.[84]

So far then we have not found convincing grounds for annexing mathematics and the fine arts. But even if we could, we still only have what I have called 'justification by association'. We would still, that is, not be justifying the learning of mathematics unless it were also shown how the arts themselves were justified. However, if we understood the value of the arts in education it might suggest the essential criterion which mathematics must have to gain parity with those arts. So, we will pause to consider a fairly elaborate rationale for art which has been put forward, to see what it is about art which justifies its place on the curriculum. Accordingly, the rationale I shall be discussing will provide the opportunity to usher in one final attempt to show that mathematics is a fine art by dint of its special properties.

[84] Some of these aesthetic aspects have already arisen in my discussion of earlier writings from Moritz's anthology. They will be revisited when I finally detach aesthetics from fine arts.

Chapter 6 – The fine arts, education and mathematics

In the previous chapter we saw that writers, both ancient and modern, have tried to make connections between mathematics and art. Although the meaning of art has undergone some change since the eighteenth century we can suppose that at least some of these writers have suggested that mathematics resembles, or even provides an instance of, fine art alongside paintings, poetry, music and the like. There have been several reasons why writers and mathematicians have come to this conclusion and these reasons seem to appear again and again. Prominent amongst them is the view that the mathematician creates rather than discovers mathematics. Also writers are often struck by the fact that mathematics has certain aesthetic properties particularly, but not exclusively, that of beauty of some kind or other. But these considerations by themselves still seem to ignore the fact that fine art has an emotional component which is absent in mathematics. However, at least one writer, Kline, has even thrown doubt on this objection. So far I have insisted that these writers are unconvincing.

But suppose that such arguments had been convincing, it is not clear what would be gained from re-categorising mathematics as a fine art. If mathematics were, after all, a fine art would this give us a strong reason for learning it? Unless we want to provide more than what I have called 'justification by association' we would still need to find a purpose for learning mathematics via the value of the fine arts themselves. I cannot of course review the many and various rationales which set out the educational value of the arts. What I shall do is to rely largely upon the work of a leading contemporary philosopher of education, John White, whose reputation has been gained largely on the strength of his pioneering and continued work in the aims of education. After considering his work, I shall examine the argument of one final writer who believed that mathematics is a fine art because it has some of the same kind of educational value that White ascribes to art.

The educational value of fine art

Several justifications for studying art are given by White (1990). By 'art' White indicates that he means paintings, poetry, music and literature, and hence not mathematics. Indeed, at one point when he does refer to mathematics it is to make something of a contrast with art. I shall firstly examine why White thinks that art is of special value in a person's education, before considering whether or not mathematics could be valued in the same way.

Clearly, if art were a form of knowledge then this alone might secure art a place in the curriculum, given that education is closely connected with knowledge. But White acknowledges that there is a problem with supposing that art can be a form of knowledge, even though he maintains that art can enlarge one's knowledge. What art does do, according to White, is to enlarge the range of options to be pursued for their own sake and 'for opening doors to further options including vocational ones'. He realises that many other activities, including the study of mathematics, can similarly extend our options. But he singles out art for special consideration because of 'the enormous delight which art can give us - so immediately and often with so little struggle - as compared with so many other intellectual or practical activities' (p.155).

The delights of Mathematics, on the other hand, are not so easy to achieve. Even those who have wanted to establish a strong connection between mathematics and art have highlighted the difficulty facing anyone who is to enjoy this aspect of mathematics. In particular, Huntley (1970) in his study of 'mathematical beauty', admits that whilst 'a limited sense of aesthetic appreciation is given; the rest must be acquired', reminding us once again that many of the enjoyable aspects of mathematics are not easily accessible. He continues:

> ... the mathematically uneducated can appreciate the dual symmetry of the ellipse; that is given. But the unlimited store of beauty of the conic sections is reserved for the mathematically trained: it is acquired. This indicates that the path to real aesthetic pleasure is through toil, a principle that holds far beyond the realm of mathematics. Spadework is essential... (p.2)

So one thing that we shall need to take up later is the extent to which the accessibility of aesthetic pleasure in mathematics is worth the effort it takes for its acquisition. We shall need to ask whether or not aesthetic pleasure can be achieved much more easily in other areas of the curriculum, and hence that these areas should be the sole or main source of such enjoyment of the kind attached to art, or, on the other hand, whether aesthetic pleasure should be sought wherever it exists. We need to know whether or not it is the specificity which counts, whether or not we can argue by analogy with protein, which vegetarians learn must be taken from foods of different kinds to achieve balance and variety.

So far, then, White argues that art can enlarge knowledge, and does enlarge options for a certain kind of enjoyment much of which is relatively accessible. But there is more to come. 'Art unlocks emotion in one' he writes, it '...breaks through the crust of our conventional way of thinking and behaving, and through our ordinary practical involvement in our affairs'. So that art 'liberates':

> We take pleasure in exercising our powers in a spontaneous, unfettered, way. In satisfying these desires so deeply implanted in us, art contributes directly to our well-being (p.155).

Something on these lines has been claimed for mathematics, as we have seen, when the creativity of mathematics is stressed, and if such a view could be sustained then mathematics at least has some properties that resemble art. But to make the strong claim that mathematics *is* art would need much more argument, particularly since creativity if it is possible in mathematics is largely achieved by the initiated.

If we continue with White's account of the importance of art in education we can see again where it diverges from mathematics. The next level of justification of the arts arises, as it were, from the other end. We delight in extending our emotional life, by giving our emotions full leash, but we also experience conflicts amongst our desires and seek some order. When all that is required is some prioritising amongst desires of different value, some ordering is possible through developing strength of will. But sometimes we experience what White calls 'irresolvable tensions between our most important desires themselves'. For example, he remarks that:

> It may be impossible for me to be both an artist and a man of affairs; to be fully responsive both to the loyalties which claim me and to my personal projects; to see the world as objectively as possible on the one hand and from my own subjective perspective on the other... (p.156)

Such a predicament provides a strong role for art. Referring to such conflicts of this kind, White believes that 'Art enables us to come to grips with them'. More specifically, he says of art that:

> ... it can do this by working on our imagination: we experience the tensions in what we see as the artist's soul, or in those of the characters he or she creates. In this way we approach our own conflicts by contemplating their counterparts in others; and here we do not merely contemplate them, but experience them within the framework of the work itself. The work contains them and enables them to co-exist within a unity, a formal structure. This helps us to reconcile ourselves to the ineluctability - we see our tensions as something we have to live with, something we can hold together within the framework of our life. A work of art comes to stand proxy for our own life. (p.156)

This awareness that art brings [85] is characteristic of the autonomous rather than the heteronomous individual and hence we may say that, in this way at least, the study of art contributes towards the development of autonomy.

As White reiterates:

> We want life, spontaneity, and wealth of sensory and emotional experience; and we want boundedness, order and framework...a work of art by embodying this tension within itself, can help us to hold these two opposites together. It shows us how it can be done.
>
> (p.157)

Others have made essentially the same point. O'Hear (1988) writes:

> In works of art and literature...we come to a particular, inward type of understanding of human life and experience. This inwardness derives from the way in which in a work of art there is a sympathetic enactment of the experience depicted or evoked, and in which, on the part of the author and audience, feeling and receptivity are at their peak (p.90).

White also outlines what he calls a more 'conservative' role for art. It seems that art allows us to reflect upon our common values, for many of which no rational argument can be found to justify. One of art's roles is to reaffirm these values. In this way it contributes towards our *self-knowledge*. So White writes:

> It is because of this affinity between art and our nature that experience of the former has a role in education which goes far beyond its value as an optional activity for those who prefer it to other things. Art is necessary for everyone, because self-knowledge - in its practical aspect - is necessary for everyone. (pp158-159)

and, again, reiterating what he had said earlier in the book about autonomy, he says :

---

[85] Russell comes quite close to making a similar claim about mathematics. Firstly he outlines the place that mathematics should have in one's life:
'What is best in mathematics deserves not merely to be learned as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement' (Moritz, 1914 p.182).
Then he goes on to give reasons why. The world of mathematics, Russell believed, can compensate us for the shortcomings of the everyday world:
'Real life is, to most men, a long second-best, a perpetual compromise between the real and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world'(Moritz pp.182-3).

> ...the autonomous people we have in mind need, for the sake of their own well-being, to be closely in touch with their own structures of desire, including their feelings; and not to be confused or mistaken about their priorities, or in other ways self-deceive. In its power to reveal ourselves to ourselves, and thereby to confirm us in what we take ourselves most deeply to be, and also in our sense that our values are not idiosyncratically our own, but shared with countless others across space and time, art is an unparalleled vehicle of self-understanding, and so of education. (p 159)

White's last line on the role of the arts is especially pertinent:

> Their intimate connection with self-knowledge and personal well-being give them a curricular importance which mathematics, say, or science could not hope to rival (p.160)

White's argument is most convincing when the art in question is literature, painting or sculpture. But it becomes difficult to see how absolute music could help us come to grips with conflicting desires in quite the way in which he outlines it. But suppose it could be argued that mathematics could contribute to self-knowledge. Then even if it could not rival the fine arts it may at least equal them. This is precisely the claim made by John William Navin Sullivan.

Mathematics and self-knowledge

Sullivan, in his 'Mathematics as an Art' (In Newman, 1956), makes the bold claims that 'Art which is worthy of the name reveals to us some aspect of reality'. A similar revelation, he believes, arises in mathematics:

> Mathematics, as much as music or any other art, is one of the means by which we rise to a complete *self-consciousness*. The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds[86] (p.2021)

Here then is a clear statement that mathematics is a fine art since it provides us with a particular kind of self-knowledge. As we have just seen, White also singled out the special value, peculiar to the arts, of *self-knowledge* [87]. However, we should note from

---

[86] My italics

[87] This aspect of self-consciousness is an important one and is not usually included by those who simply want to assert that mathematics has an aesthetic aspect. It strengthens my resolve to distinguish between those things which are aesthetic, and those things which are art either in addition to being in the aesthetic realm or art without being aesthetic.

the start that not only is self-knowledge only one of several values of art which White discusses, but also that Sullivan's notion of self-knowledge is much more restricted. For White, self-knowledge includes knowledge of common desires and feelings, whereas for Sullivan it appears to be solely knowledge of the relationship between mind and the external world.

Much of Sullivan's main argument is similar to Kline's, (see above page 83). He begins from the historical fact that although mathematics was once regarded as providing absolute knowledge of the external world, through the rarely questioned truths of space given by Euclidean geometry, it gradually became evident that alternative, consistent geometries could be created. So he notes that 'the mathematician is entirely free, within the limits of his imagination, to construct what world he pleases'. However, the connections between this created mathematics and what we might call the external world, but which Sullivan simply refers to as 'experience', presents him with a problem. He rightly says:

> If he [the mathematician] can find, in experience, sets of entities which obey the same logical scheme as his mathematical entities, then he has applied his mathematics to the external world; he has created a branch of science (p.2020)

However, he is clearly not satisfied with the idea that the mathematician has hereby simply created 'science', rather than art. So he invokes the creativity aspect once again saying: 'Since...mathematics is an entirely free activity, unconditioned by the external world, it is more just to call it an art than a science'. But there is a price to pay for this realignment: mathematics-as-an-art cannot, it seems, give us the kind of knowledge that Euclidean geometry once did. The moment we use created 'mathematical models' for understanding the external world, we are engaging in science. Many would agree that this is indeed the state of affairs with respect to pure mathematics and science. But perhaps because the creative nature of mathematics does not seem sufficient for classifying mathematics as an art, Sullivan goes much further and boldly asserts that:

> Art which is worthy of the name reveals to us some aspect of reality. This is possible because our consciousness and the external world are not two independent entities. (p. 2021)

He has therefore assimilated mathematics into art only by taking on something rather like an idealist position, in order to side-step the dualism implicit in mathematics as it is applied in science. From then on he reiterates his earlier point:

> Mathematics, as much as music or any other art, is one of the means by which we rise to a complete self-consciousness...the real function of art is to increase our self-consciousness; to make us more aware of what we are, and therefore of what the universe in which we live really is. (p.2021)

But to say that mathematics provides essential self-knowledge, and to say that art does this too, is not sufficient to show that mathematics is art. This would be to commit the fallacy that If A is S and M is S, then M is A. Nevertheless, in his concluding lines this is precisely what Sullivan does:

> ...it is certain that the real function of art is to increase our self-consciousness; to make us more aware of what we are, and therefore of what the universe in which we live really is. And since mathematics, in its own way, also performs this function, it is not only aesthetically charming but profoundly significant. It is an art, and a great art. It is on this, besides its usefulness in practical life, that its claim to esteem must be based. (p.2021)

There is therefore something *formally* wrong with Sullivan's argument. But even if we could accept that a certain kind of self-knowledge was a necessary and sufficient condition of the arts, it is not clear that this would clinch the matter. As I have already mentioned, the particular *kind* of self-knowledge, which Sullivan believes that mathematics can provide, is much narrower in scope than that which is obtained from the arts. To see how this might rule out Sullivan's way of including mathematics amongst the fine arts, it is worth considering how certain philosophers have distinguished art from other enterprises. In a manner analogous to Sullivan, Schopenhauer (1819) suggests that art is similar in one respect to philosophy since both are concerned with the question 'what is life?' But he does not thereby assimilate art into philosophy. Indeed, he goes on to distinguish them, by remarking that:

> [But] all the arts speak only the naive and childish language of perception, not the abstract and serious language of *reflection*; their answer is therefore a fleeting image: not permanent and general knowledge...their answer, however, correct it may be, will yet always afford merely a temporary, not a complete and final, satisfaction. For they always give merely a fragment, an example instead of the rule, not the whole, which can only be given in the universality of the *conception.* (See Hofstadter and Kuhns, 1964 p.452)

Mathematics of course does not provide answers to questions such as 'what is life?' but like philosophy it deals almost exclusively with propositions of a general or universal nature and hereby lies its power, its capability for widespread application. Indeed Sullivan highlights this point when he says:

> Mathematics is of profound significance ...because it exhibits principles that we impose. It shows us the laws of our own being and the necessary conditions of experience (p.2021)

But in saying this he draws a particular kind of wedge between mathematics and art.

More recently Anthony O'Hear, (1988) in *The Element of Fire: Science, Art and the Human World,* has like White explained the value of the arts in terms of a particular kind of self-knowledge, claiming that 'in works of art and literature... we come to a particular inward type of understanding of human life and experience'. He, too, makes a similar point to Schopenhauer that such understanding is brought about 'in a concrete way, through particular experience by reference to the concrete particularity of things'. So once again we find that any self-knowledge that mathematics brings is in strong contrast to that which is provided by the arts.

There are difficulties, then, with Sullivan's arguments. Even if it is true that mathematics has a creative element, we have seen earlier (see page 71) that this is not sufficient to show that mathematics is an art, or at least an art in the full-blown sense that Sullivan has in mind. Mathematics may indeed give us self-knowledge but this is only of a restricted kind and is only achieved at the expense of adopting an undefended idealism. The self-knowledge of the kind that White has claimed art can provide, is one which helps to brings us to terms with conflicts of desires and values, on the one hand, and confirms our deep-seated common values, on the other. It is doubtful, to say the least, whether the awareness of our freedom to construct models of reality similarly resolves certain conflicts of desire. We must also acknowledge that Sullivan's argument contains a formal difficulty in the last extract above. Even if the ability to increase self-consciousness is a necessary condition of art, it need not be a sufficient one. So that even if mathematics has this property it does not imply that mathematics is thereby art. In coming to see that we can construct systems, that these are not simply given, we surely can learn *something* about ourselves. But to admit this, is not to go as far as to say, with Sullivan, that mathematics is therefore art.

Once again we have found that a strong claim that mathematics ought to be regarded as fine art cannot stand up to analysis. It is time now to ask whether it is not possible to show that *every* argument is doomed to fail, not simply on a formal point but conceptually. We have just seen how one philosopher of education has found a fundamental criterion which distinguishes art from philosophy. If we delve further into O'Hear's arguments, and adapt them slightly, we find that an additional, powerful distinction can be made between mathematics and the fine arts which shows us conclusively that mathematics cannot be regarded as fine art.

Art and Mathematics – a possible dividing line.

In *The Element of Fire: Science, Art and the Human World* O' Hear (1988) sets out a strong contrast between art and science, particularly physics, in order to emphasise the importance of art. If such a contrast as O' Hear makes could be shown to be applicable to mathematics and art, and the connection between physics and mathematics is in many places a very close one, then this will give us good reasons for supposing that no attempt to classify mathematics as art can be sustained.

Much of O' Hear's contrast rests on the notion of human perception. A significant part of his argument goes as follows: human perception is central to the arts, and is treated there as an end in itself, whereas, for science, the way things seem to be is never sufficient. Indeed an important aspect of science is its claim that common sense perception of the world is illusory. The object that concerns science is the world-in-itself rather than the world as-it-is-perceived. So that even if science begins from human perception, it necessarily transcends this in order to attain some kind of independent view, unbiased by the way humans contingently are, and aims at asserting information about how things are in themselves.

Our seeing something as coloured is, according to a long tradition of science, just the effect of which some ultimate colourless material is the cause. So that our perception of colour cannot be the basis of our understanding of the nature or properties of such material. For such an understanding we must somehow transcend the standpoint of the perception. Tactile perception has been questioned by scientists too; the famous remark by Edison that the common sense view of a table, as something which is solid, is not the physicist's notion. More recently, in quantum mechanics we learn that even the position of particles is not at all as simple as one would imagine. So as science develops there seems to be no end to the measures it will go to, to diminish the importance of human perception.

Without giving further details of O' Hear's argument at this stage, I should like to look at one aspect of mathematics in a similar way to that in which he looks at physics and consider the status that mathematics has under the demarcation that arises from his argument. It will be crucial to know whether mathematics falls on the physics side, as it were, or on the arts side. But we shall have to tread carefully. Although there seem to be close connections between mathematics and, say, physics, there is at least one essential difference between the two of them, namely, that science sets out to discover causes. So in one important aspect mathematics cannot be grouped with science in the way that O'Hear has in mind. Yet, his insistent complaint that science too readily believes it can ignore human perception does still have echoes in mathematics. We learn, for example, that a point has no size, only position, and that a line is a series of points, so that neither can be perceived. The same is true of numbers. Notwithstanding a thoroughgoing empiricist view of the nature of mathematical objects, numbers are not the properties of objects and hence cannot be perceived.

If perception is not the usual source of mathematical insight, widespread use is made of so-called 'intuition', and it seems to me that intuition does play an analogous part in mathematics to the part that human perception plays in the sciences. That is to say, just as science might be said to begin from perception, mathematics may begin from intuition.[88] We have seen that for science human perception is problematic; the question to ask now is whether intuition in mathematics has similar difficulties.

It should be pointed out that the concept of intuition is a broad one. Davis and Hersh (1980) offer six different, though related interpretations of the concept of intuition. Intuition they suggest can either mean ' the opposite of rigorous'; 'visual'; 'plausible; incomplete; or convincing in the absence of proof'; 'relying on a physical model, or on some leading examples'; or finally 'holistic or integrative as opposed to detailed or analytic' (pp. 391-2). The sense in which intuition leads to propositions which are 'plausible or convincing in the absence of proof' is a particularly important one to consider. Thus described we can enquire whether intuition must at some point part company from mathematics in a similar way to which human perception departs from physics, but not from the arts. If intuition is at some point dispensable, then we have a powerful way of removing the pretension that mathematics may be classified as art.

---

[88] For some mathematicians intuition provides the foundation of mathematics.

We do not have to look far before we find examples of the inadequacy of intuition as a means of reaching mathematical truth. Several examples of what he calls the 'failure of intuition' are discussed by Hans Hahn in his 'The Crisis in Intuition' (In Newman, 1956). He shows quite conclusively that there are many occasions where what seems to be intuitively true turns out to be false through the application of mathematics. As he writes:

> It was believed that a curve must possess an exact slope, or tangent, if not at every point, at least at an overwhelming majority of them. (p.1962)

But Hahn goes on to discuss the famous blancmange curve, by applying rigorous mathematical techniques to which it can be shown to have no tangent anywhere. The similarity of his example to O' Hear's example of the 'vault of heaven' is striking:

> One stands, let us say, on a clear night beneath a moonless sky in the Mediterranean, and the sense that there is a vault above one filled with stars is overwhelming. One *knows* that there is no such thing, that the sky and heaven's vault are illusions...(pp.8-9)

In both cases what seems to be true *from a human point of view* turns out to be false within disciplines that somehow go beyond such a human perspective.

Another example from mathematics lies in the theory of transfinite numbers. Intuitively, we might say, there are fewer even numbers than there are whole numbers. Yet Cantor has shown that if we apply the same criterion to infinite sets, as we do for comparing finite sets, then the set of evens and the set of whole numbers are the same size. The proof is simple. Create a mapping from the set of whole numbers to the set of evens in the following way: map 1 to 2, 2 to 4, 3 to 6 and in general n to 2n. The mapping is a one-to-one correspondence and hence we can conclude that these sets are the same size! Once again intuition leads us one way but mathematics leads us away from such intuition. As I take it, O'Hear believes that art never overrides human perception in the way in which mathematics overrides intuition in such examples as I have given. To the extent that this point is criterial, mathematics can never aspire to art.

So, by using O'Hear's argument, and drawing at least one parallel between mathematics and science, an important distinction between art and mathematics can be made which suggests that they are radically different in an important respect. O'Hear writes:

> What modern science aims to do is, in the name of a wider objectivity of view, to displace the human being and his modes of perceiving the world from the centre of the picture, and to present the human being and his modes of perceiving the world as incidental parts of the picture. From this point of view, the human being is seen as part of a wider and more inclusive causal process, and, from the point of that view, of no more significance than any other incident in the development of the cosmos. (p.15)

What I am suggesting is that mathematics can be seen as having a similar aim to science, even though this aim is perhaps not so far-reaching, since there is not so much of a common sense view of intuition in mathematics to override. All the same, once we turn our intuition onto mathematical objects, a similar kind of correction must sometimes be made to such intuition as it is to sense perception in science. This is in strong contrast to the way matters are in art and gives us an important reason for seriously questioning whether mathematics could ever be art in any thoroughgoing way.

Of course, if mathematics is not a fine art, nor could ever become one on the strength of these considerations, there is nothing to stop artists and mathematicians from *denoting* it as such. Nor is there any reason to suppose that mathematics may not be a source of aesthetic satisfaction, unless of course 'art' and 'aesthetic' are synonymous. It is these alternatives that I want to discuss in the next chapter.

Chapter 7 – Mathematics and Aesthetics

Art and the aesthetic are not necessarily co-existent

Mathematics, we have seen, is essentially different from the arts because it must always at some point eschew the human point of view. But even if mathematics can thus never be considered to be fine art, it may still have those aesthetic properties, which so many writers have insisted upon, unless, of course, we *equate* 'aesthetic' and 'art'. Both Sullivan and Kline in their attempts to show that mathematics is fine art have implied a distinction between the fine arts and aesthetics. Kline found it necessary not only to invoke aesthetic requirements in the origination of mathematics but also went on to discuss the extent of the feeling side of the arts and how that could be squared with mathematics. Sullivan is even more explicit in the way he shows that the existence of aesthetic properties is not a sufficient condition of the fine arts. Before setting out his account that mathematics is a fine art, which we have already seen is inadequate, he wrote:

> The literature of mathematics is full of aesthetic terms, and the mathematician who said he was less interested in results than in the beauty of the methods by which he found the results was not expressing an unusual sentiment... But to say that mathematics is an art is not to say that it is a mere amusement. Art is not something which exists merely to satisfy an "aesthetic emotion".
>
> (In Newman, 1956 pp. 2020-2021)

Art for Sullivan, then, is something more than mere aesthetic delight, nevertheless he does acknowledge and distinguish the presence of something which at least provides 'mere enjoyment'. To refute Sullivan's claim that mathematics is a fine art, as I have done, is, therefore, not to deny that it may still be a source of 'aesthetic emotion'. Moreover, if mathematics is a source of aesthetic satisfaction this might provide some justification for studying the subject. Both Kline and Sullivan are suggesting that art and aesthetic are not *co-existent*. If they were *co-existent*, then to cast doubt over the connection between art and mathematics, as I have done, would at the same time be casting doubt on the possibility of the aesthetics of mathematics. Other writers, too, have been explicit in saying that whilst art and the aesthetic are not necessarily mutually exclusive they are independent of each other. If this is the case, then the question of whether mathematics has an aesthetic aspect, and whether this in any way can provide a justification for learning it, still remains unanswered by inadequate arguments to show that mathematics is fine art.

Without the coexistence of art and aesthetics, any object may, in principle, fall into one of four categories, according to whether it is deemed:

1. to be fine art but without aesthetic properties.
2. to have aesthetic properties (or be a source of aesthetic experience) but *not* be fine art.
3. to have aesthetic properties (or be a source of aesthetic experience) *and* be fine art.
4. to have *neither* aesthetic properties (nor be a source of aesthetic experience) *nor* to be fine art.

As far as mathematics is concerned my argument so far rules out the possibility of mathematics falling under category 1 and 3, but it still leaves open the possibility of its falling under category 2 (or even category 4).

It must be admitted that there is a sense in which aesthetics has come to mean the Philosophy of Art, in the same way as ethics has become the Philosophy of Morality. So that the connection between aesthetics and fine art may be regarded as a necessary one. Yet the examples from natural objects do show conclusively that not all that is aesthetic is art, if by art we primarily mean *artefacts*. So it is at least conceptually possible to claim that mathematics is aesthetic without thereby entailing that it is art.

It is often pointed out that the word 'aesthetic' was introduced in the eighteenth century by Alexander Gottlieb Baumgarten who adopted the Greek word for perception to denote the study of perception in contrast to the study of what is known. He collected the study of art under the notion of aesthetics and the two were seen as identical. Strictly, speaking then, aesthetics is an enquiry about a special kind of experience, typically, but not exclusively of art. It is particularly attached to beauty and modern writers are still concerned to elucidate this kind of experience.

Aesthetics, then, is not a sufficient condition of art. But Binkley (1977) makes it clear that aesthetics need not be a necessary condition of art either, so that all four of the categories in the above list are indeed possible. He writes:

> An artwork is a piece. The concept "work of art" does not isolate a class of peculiar aesthetic personages. The concept marks an indexical function in the artworld. To be a piece of art, an item need only be indexed as an artwork by an artist. Simply recategorizing an unsuspecting entity will suffice (pp37-38)...There are many kinds of "pieces", different according to the practices they are indexed within. A "piece" could be a piece of *mathematics*, or economics or art; and some pieces may be addressed to several disciplines. An artwork is just a piece (of art), an entity specified by conventions of the practice of art.[89] (p.39)

His notion that art is simply that which is indexed as art by an artist is a helpful one for explaining much of contemporary work found in exhibitions. He discusses in detail one such example: a representation of the Mona Lisa who is given a moustache. His main point is that *perceiving* this work is unimportant since there is nothing special about the way the moustache is drawn. One could come to know the work just as well by a description. Sense perception is no longer essential in coming to grips with this work since it is not the *look* of the work that matters.[90] The same could be said presumably about Cage's 'silent' piano piece.

Binkley concludes from this that:

> ... "Is it art?" is a question of little interest. The question is "So what if it is?" Art is an epiphenomenon over the class of its works. (p.38)

What it is very important to note, however, is that Binkley's account rests upon a nominal account of art. To index something as art is more or less to say that it is art. Of course not *anyone* can say this, but even so it does leave the question of what can be deemed art as a very open one; no determinate properties of a work are necessary for it to be indexed as art. Binkley, himself, admits that a 'piece' of mathematics could be deemed as art in this way. In particular, mathematics may be indexed as art without implying that it has any aesthetic properties. Binkley clearly thinks that the question of whether mathematics is art would be of little interest. For us, however, it is crucial, because, as I am suggesting, by association with art mathematics is supposed to have special educational value. So *if mathematics is indexed as art the value of mathematics-as-art will be of crucial interest in deciding whether or not this way of regarding mathematics has any justifying force*. Binkley does not provide us with any concomitant value with the indexing of objects as art. If art has no special educational value after all,

---

[89] My italics

then on this view of the matter it is pointless to try and decide whether the connection between mathematics and the fine arts is a close one.

Leaving aside the question of whether we should simply *index* mathematics as fine art, I now want to explore the other option which has arisen from this discussion. Given that aesthetics and the fine arts are not co-existent, I want to examine the various ways, if any, that mathematics could be included under category 2 in the above list. That is to say I want to determine how it might be deemed to have aesthetic properties (or be a source of aesthetic experience) even though it is not fine art.

Mathematics as a source of aesthetic experience

The possibility that mathematics can be a source of aesthetic experience has been put strongly very recently by Morgens Niss (1994) who has claimed that mathematics is a *field of aesthetics*. He follows a range of writers in the present century who have argued that mathematics has special properties which are a source of special enjoyment. But this rather general claim needs to be examined.

One way of understanding the notion of 'aesthetic' is to regard it as involving a particular kind of *experience*. Since at least the time of Kant, the notion of *disinterest* has become a central defining characteristic of such experience. Disinterest, in the aesthetic context, as elsewhere, is not contrasted with *un*interest but rather with certain kinds of partiality associated with *non*-aesthetic experience which will be outlined presently. But even if theorists are right in describing aesthetic experience in this way the task remains of establishing whether opportunities for such experience exist in mathematics.

It must be pointed out that few *aestheticians* seem to admit readily that mathematics is a source of aesthetic experience. But many point out that the range of objects about which aesthetic *judgements* can be made is indefinite, even though works of art and features of the natural world are amongst the most typical. The following passage by Charlton (1970), illustrates this point well:

---

[90] In this respect, of course, Binkley's view is in contrast to that of O'Hear.

> You are thinking of taking a flat: you look it over, and note, among other things, that the windows are well or ill proportioned. You are driving somewhere on business, and see that the countryside has changed in character, become more sombre and severe. You are trying on clothes, and consider how they look on you and how they make you look. You are listening to a talk, and contrast the banality of what the speaker is saying with the pleasant tones and well chosen words in which he says it...You are bedding out some plants and try to get the spacing even or the colours well grouped ... it strikes you that the postman has a funny face...*aesthetic judgement is exercised most formally, perhaps in art galleries, concert halls and the like, but our lives would be very dull if it was in perpetual abeyance outside these temples dedicated to the muses.*[91] (pp. 9-10)

Stolnitz (1960) gives a clear, though not uncontested, account of aesthetic experience, which he prefers to call the aesthetic *attitude* but which is nevertheless characterised by disinterest. Moreover as we shall see he does make explicit and helpful reference to mathematics as an appropriate object to which one can adopt the aesthetic attitude. Stolnitz remarks that most of our perception is of a practical kind whereby we notice only the features of things which serve our own purposes. It should be clear that the attention we pay to mathematics is usually of this kind especially when it is presented as a tool, a set of techniques, which can be used to solve practical problems. Yet we do not always perceive things in this way. There are occasions when as Stolnitz puts it '...we pay attention to a thing simply for the sake of enjoying the way it looks sounds or feels'. On these occasions we have adopted what he calls the 'aesthetic attitude' of perception, which he defines as 'disinterested and sympathetic attention to and contemplation of any object of awareness whatever, for its own sake alone'(p.19).

It should be pointed out that the kind of account that Stolnitz gives is one where aesthetic experience is defined negatively; it is experience which does not have certain features. Thus Stolnitz gives a few examples of cases which are not characterised by disinterest. They are the cases where one wants primarily to own, gain knowledge about, or pass judgement on, the object of one's attention. So if we are to adopt this special aesthetic attitude towards mathematics it must be possible to avoid attending to certain of its objects in at least these ways. In particular we must not be attending to mathematical objects in order to learn something about them. We must somehow be just attending to certain features and enjoying the way mathematics sounds or feels.

---

[91] My italics

But mathematics is characterised as an abstract discipline and, as such, it does not have 'a look', 'a sound' or 'a feel'. Mathematical objects are not perceived through sense experience. So it is difficult to see how certain apparently necessary conditions for aesthetic experience could be fulfilled in quite the way that Stolnitz outlines, even if our attention were of a non-practical kind. But Stolnitz does entertain the idea that mathematics is an object towards which we may appropriately adopt an aesthetic attitude. As he explains:

> There is another kind of "awareness" that occurs, though relatively infrequently, in adult experience. This is "intellectual", nonsensuous knowledge of "concepts" and "meanings" and their interrelations: such knowing takes place in abstract thinking, such as logic and *mathematics*. Even if images or "pictures" accompany such thinking, they are only secondary. When the mathematician thinks of the properties of triangles, his thought is not restricted to any particular triangle he may "see in his head" or draw on paper. A man who develops a system of mathematical logic is occupied with logical relationships which are neither sensed nor perceived. Now this kind of apprehension can also be aesthetic. If one's purpose is not, for the moment, problem solving, if he pauses to contemplate disinterestedly the logical structure before him, then his experience is aesthetic…to take account of such experience as well as sensation, I have used the broad term "awareness" rather than "perception". *Anything at all, whether sensed or perceived, whether it is the product of imagination or conceptual thought, can become the object of aesthetic attention.*[92] (pp. 26-27)

Often, as Stolnitz indicates, aesthetic experience is linked with a contemplative stance. This is true provided we do not suppose that this always implies a minimum amount of intensity. There are degrees of intensity in aesthetic attention. As Stolnitz points out: 'A color, briefly seen, or a little melody, may be apprehended on the "fringe" of consciousness, whereas a drama will absorb us wholly'. Aesthetic experience may not be immediate either ; Stolnitz reminds us that we may have to walk round a sculpture or through a building like a cathedral to appreciate it. The same is true of aesthetic appreciation of certain mathematical objects like proofs, as we shall see. Finally Stolnitz reminds us that aesthetic awareness often involves taking notice of detail and we may need to have this detail pointed out, or even have a certain amount of technical

---

[92] My italics. Stolnitz seems to think like many others that this experience is an adult experience. Compare this with what Leibeck, 1984 has to say.

training. This will be particularly true of mathematics where, as I have already remarked, the aesthetic awareness is not of something sensual.

We do have good reasons, then, from at least one aesthetician, to suppose that it is possible to have aesthetic experience in mathematics, even if such experience must involve a particular attitude. Moreover, this experience is markedly different from that which we are expected to have in realising the more familiar purposes of the subject. Furthermore, the experience might be difficult to achieve, indeed it may be scarcely accessible to children, and it may or may not be as intense in mathematical contexts as it is elsewhere. So we shall need to examine carefully the objects within mathematics where opportunities for such experience are supposed to exist. One particular feature which mathematicians claim is to be found almost everywhere in mathematics is *pattern*. So it is to a consideration of this aspect that I now turn.

Formalism and the ubiquitous pattern

In one particularly influential theory of aesthetics it is *formal* properties that are regarded as the most important criteria of aesthetic evaluation and the source of aesthetic pleasure. Subject matter, and any expressive properties of an object are subordinated to the relationships between its parts whether these are colour, sound, textural structure of words or, in considering the parts together, what many formalists call *unity*. An aesthetician whose work is worth considering in more detail is that of Charlton, since he discusses in some detail the formal notion of a pattern.

We should note from the start that Charlton's notion of a pattern is in one sense very general and in another sense very specific. It is general in the sense that it is defined as 'a change through space in respect of visible qualities'. What kind of change in visible qualities is left open, at this stage. So that *any* configuration of visible qualities may be a pattern, though of course it might not be a particularly enjoyable one. On the other hand the fact that, for Charlton, the change must be one of *visible* qualities restricts the context in which patterns can exist. They cannot on this account exist in music, let alone mathematics, since music is essentially an aural experience. Indeed Charlton describes music and patterns separately and makes comparisons between them. Whereas a pattern is a 'a change through space in respect of visible qualities' music is 'a change in sound which goes on through time'.

Having distinguished music from pattern Charlton goes on to enquire how each can be *enjoyable* since not any visible or aural change will be appealing. He therefore seeks to explain what a *good* pattern is, or as he prefers to say, a 'successful' pattern. There are parallels it seems between a successful pattern and a successful piece of music. Both involve *following* a change. As Charlton suggests:

> ...listening to music is trying to follow the change in sound, and ...music has merit by formalist criteria if it is a change which is easy for people who are in general able to follow music, or music of that type, to follow attentively. (pp. 40-41)

A little more of what is involved in following music is given when he says: 'It seems that anyone who is following any sort of process must know what is going on at any moment'. During the changes in sound, which as we have seen are criterial to music, Charlton insists that 'the listener should know where he is in the change at any moment...must be able to tell...at what intervals the notes are...' This is necessary because it is important to see that these notes are *related* and also since success involves achieving *unity*, an important mark of aesthetic excellence. He also adds:

> When we enjoy an activity, not only is what we do determined by factors internal to the activity, but those factors seem by themselves a sufficient reason for doing it. If listening to a piece of music is to stand a good chance of satisfying this requirement, the piece must contain variety, surprises, even some difficulty. (p.45)

If we now try to explain successful pattern on similar lines we seem to have a difficulty at the outset. Whereas, music has quite definite pitches, timbres and rhythms, this is not true of colour, which provides part of the visual material of pattern on Charlton's account. However, *size and shape* are visible qualities about which we can be precise. So Charlton builds these features into his account of the enjoyment of pattern:

> If a pattern is to have unity, it must be clear that the elements in it are related...there is only one way possible: quantitatively...a pattern has unity insofar as the beholder can tell how great each distance, angle etc. is, not absolutely, but in relation to the others. That is once again, the pattern must be followable. (pp.49-50)

But size and shape are mathematical properties and thus it is these which underlie Charlton's notion of pattern. We seem to be back with a view similar to Plato's and, in particular, to that of St. Augustine. That is to say, on Charlton's account, for a pattern to succeed in being enjoyable its mathematical properties must be followable. Moreover this idea of a pattern's being followable means that for Charlton enjoyment is achieved rather more 'discursively' than 'synoptically'. He writes:

> Aestheticians who advocate formalist criteria often contrast *'discursive'* with *'synoptic'* perception, *going over and connecting up the parts of an object* of awareness with *contemplating it as a whole*; and they say that the former is characteristic of scientific observation and practical life, while the latter is typically aesthetic. Insofar as the distinction is valid...I have urged just the opposite. *Contemplating a pattern synoptically* or all at once would be contemplating it as the end of a temporal process, and if we are to enjoy it we must rather contemplate it discursively as a spatial process...On the other hand, even if the scientist or practical man is to observe some part of what is before him, he must be aware of it as some sort of unified whole or at least through awareness of such a whole.[93] (pp.52-53)

As we shall see there are patterns in mathematics, particularly but not exclusively number patterns, for which this discursive approach of following quantities seems possible, but which need not underlie any visible properties. Indeed it is tempting to suggest that mathematical patterns are rather like music in this respect since they exist as *possibilities*, as formulae which may be 'written out' in a similar way to the way in which a composer sets out instructions for the possibility of a performance. But there is at least one major difference between mathematics and music in this respect. Whereas the composer, like other fine artists, 'works out', as Charlton puts it, his or her own patterns in composing music, it is not clear that mathematicians originate their own patterns in this way.

Mathematicians not only assert that mathematics embodies or contains pattern. Some, like W.W. Sawyer (1955) for whom mathematics simply *is* 'the classification and study of all possible patterns' (p.12), even define the subject in terms of such entities. But does the mathematician, somehow 'work out' patterns just as Charlton says that some practitioners do? To answer this we need to distinguish between 'working out' in the sense of *originating* a pattern from scratch, as it were, and 'working out' in the sense of discovering a pattern amongst chaos, or perhaps identifying and describing an already existing pattern. The first of these is the stronger case and would be typical of an artist who, as I have discussed earlier, is often characterised as one who *makes* something. So it would be quite reasonable to suppose that a painter, architect or sculptor works out a pattern from scratch and that such a pattern might be just as Charlton describes it 'a change through space in respect of visible qualities'. But whether a mathematician can originate patterns seems at first sight to be highly questionable. Yet this is not an opinion which is universally held. G. H. Hardy, one eminent mathematician who was in no doubt about the importance of mathematical pattern, wrote:

> A mathematician, like a painter or a poet, is a *maker* of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas...beauty is the first test: there is no permanent place in the world for ugly mathematics.[94] (Hardy, 1967 pp.84-5)

So whilst it may still be doubted whether a mathematician *makes* anything at all, except perhaps discoveries, for Hardy at least, a mathematician does make patterns, the chief criterion for the success of which is beauty. Whether mathematics is in fact created or discovered is a difficult question and is one that I touched on earlier in my discussion of creativity. In a sense it is true that mathematicians do construct axiomatic systems, but this does not seem to imply that they can claim to have thereby constructed the patterns which are contained within those systems. Popper (1972) very convincingly wrote: 'The series of natural numbers which we construct creates prime numbers – which we *discover* – and these in turn create problems of which we never dreamt'. (p 138). The same, it seems to me, can be said of pattern. Even if we admit that mathematics is created - or perhaps more neutrally that it is constructed - there is no need to suppose that the particular patterns which are later discovered are those which mathematicians had ever dreamt of.

Aesthetic experience in discovering existing patterns

What is clear, then, is that within the number system, at the very least, patterns exist in terms of quantities. Moreover it is possible to enjoy the discursive activity of following these patterns, many of which are fascinating, some of which are even surprising perhaps. It is thus appropriate to adopt the aesthetic attitude towards mathematics to the extent that the pattern may be contemplated, or followed discursively, independently of any practical value that they might have. So that certain sets of numbers, independently of any use which is made of them for practical purposes, may themselves be regarded as constituting pattern. For example: 2,4,6,8... and 1,2,4,8,16...

---

[93] My italics
[94] My italics

For whatever reason the numbers 1, 2, 3, 4... together with certain operations for combining them became known as the system of 'natural numbers' and this name is rather appropriate. Patterns are revealed or may be discovered within the system in a way analogous to the *natural* beauty of landscapes in nature. An example of such a pattern is to be found amongst those numbers that cannot be expressed as a consecutive sum of natural numbers. Most numbers can be expressed in at least one way as a consecutive sum:

$6 = 1 + 2 + 3$

$23 = 11 + 12$

$94 = 22 + 23 + 24 + 25$

However, as it turns out none of the numbers in the sequence 1, 2, 4, 8, 16... can be expressed in this way. This sequence of numbers forms a simple pattern. Although it may have no apparent practical application it is clearly followable and contains an element of surprise when it is first discovered. It is thus likely to be a source of pleasure or satisfaction as it is being followed, or perhaps when it is contemplated as a whole. Such enjoyment provides at least an explanation for engaging in mathematics of the kind raised earlier by Wells and Leibeck (see page 37). So we have the possibility of aesthetic satisfaction simply at the exploratory stage of mathematics.

Mathematicians are seldom content with exploring patterns. They usually seek a generalisation, and also a proof that the observed pattern remains so indefinitely.[95] If such a proof is forthcoming then a result within the system is discovered. If the result can be used for an extra-mathematical situation then we can say that the study of pattern has both an aesthetic and a practical outcome. One important way in which a mathematical result can be said to be practical, is when it can be used to correspond to certain significant features of an extra-mathematical situation. So that it provides a mathematical model of such a situation about which information can be established indirectly.

---

[95] I am thinking here of pattern found in *infinite* sequences. The number of terms in a cyclic pattern is finite. Nevertheless, explanation of another kind may still be sought for these. For example, one might want to explain why some fractions are recurring decimals and others are not.

A simple example can be given of these stages, using once again the natural numbers and an anecdote based on an episode in the childhood of the brilliant mathematician Gauss. At school Gauss's teacher had asked the class to find the sum of the first 100 natural numbers. Gauss answered almost at once '5050'. He had seen that:

$$(1 + 100) = (2 + 99) = (3 + 98) = \ldots = (100 + 1) = 101$$

So that:

$$(1 + 100) + (2 + 99) + (3 + 98) + \ldots + (100 + 1) = 100 \times 101$$

$$= 10100$$

But the sum above uses the first 100 natural numbers exactly twice and hence 10100/2 = 5050 is the required sum. The use of pattern allowed Gauss to convert a lengthy sum into a simple product.

The series of numbers used above, 1, 2, 3...100, is called an arithmetic progression and it can be seen that Gauss's observation can be generalised in order to find the sum of any arithmetic progression. The defining feature of this progression is the 'constant difference' between its terms. Thus *n* terms of the progression can be expressed in algebraic shorthand as:

$$a,\ \ a + d,\ \ a + 2d,\ \ a + 3d,\ \ \ldots\ a + (n - 1)d$$

By adding the first and last terms we get $2a + (n - 1)d$. But also by adding the second term to the penultimate term we get: $(a + d) + (a + (n - 2)d) = 2a + (n - 1)d$ which is the same as before. It is not difficult to see that if we add the third term to the term immediately before the penultimate one the same result is achieved. Moreover, we can continue in this way for as long as we wish and the result $2a + (n - 1)d$ will remain the same for the simple reason that as we always gain one 'd' from the first term in the sum we lose a 'd' from the second term. Clearly, if we continue *n* times to add pairs of terms in this way, like Gauss, we will have added together the *n* terms of the sequence twice since each term has been used twice. If therefore we halve the result we will have an expression for the sum of *n* terms of an 'arithmetic progression'. Algebraically, this is written $n(2a + (n - 1)d) / 2$.

Now, arithmetic progressions of this kind are of practical use precisely because an awkward sum is converted into a manageable product. For example, if I save £5.00 of my earnings one month, £5.50 the next month, £6.00 the next, and so on, how much will I have saved in, say, 5 years? This looks difficult but once it is noticed that the amounts saved follow an arithmetic progression then a correspondence can be set up with relevant features of the problem and the algebraic result determined above. Clearly, 'a' must correspond to 5, 'd' to 0.5 and 'n' to 60, so that we have:

> $60(2(5) + (60 - 1)(0.5))/2 = 30(10 + 29.5) = 1185$ and hence I save £1185 pounds.

Aesthetic experience in producing mathematical arguments.

The proof that has been given here has been based on the ideas of Gauss, and when written more concisely it is usually regarded as being *elegant*. It is somehow pleasing in the way it reveals a truth. Moreover, although such proofs are in a sense discovered, they are also constructed, and aesthetic ideals are applied in their construction; elegant proofs are preferred to inelegant, rambling or obscure proofs. Notice therefore that aesthetic satisfaction can arise not only from *discovering* certain features, or perhaps *uncovering* such features, since this seems to include the discursive approach to pattern-following which I am adopting, but also from creating or, again, more neutrally, from *constructing* proofs in certain ways. In general we can refer to this as *producing mathematical arguments*.

If this distinction between different sources of aesthetic satisfaction is not convincing, then another explanation might help. Once a pattern has been found in mathematics no more 'finding' seems to be required, nor is the way in which it was found of much importance. The pattern has a completeness that cannot be improved upon. On the other hand, the number of proofs that may be constructed to establish a mathematical result is indefinite. (One only has to consider the number of proofs of the theorem of Pythagoras) Some of these theorems will be deemed more elegant and, therefore more aesthetically pleasing, than others.

One writer of a mathematical textbook, which sets out to teach approaches to proving mathematical results, has urged the student to contemplate the finished item as an object of admiration. Velleman(1994), in likening the construction of a proof to the quite haphazard ways in which a jigsaw is completed, writes:

> When you finish a jigsaw puzzle, you don't take it apart right away, do you? You probably leave it out for a day or two, so you can admire it. You should do the same with a proof. You figured out how to fit it together yourself, and once it's all done, isn't it pretty?
>
> (pp. 82-83)

I shall give an example of certain proofs which I recently used with able primary school children. The proofs in question concerned the consecutive sums discussed above. A proof of the general result is quite difficult, but the proof that any odd number can be made in this way is quite easily constructed. One proof was given as follows: if you halve an odd number you get two identical numbers with halves. By taking the half off one and adding it to the other a consecutive pair is formed. Not all the children were happy with this explanation and I offered an alternative: subtract *one* from any odd number and you must get an even number. Halve the even number into two identical whole numbers, then add the subtracted *one* to one of these identical numbers and once again you have a consecutive pair. Whether or not the children did in fact prefer this is an empirical matter, but what is clear is that the second proof is still arguably the more elegant. It is more elegant because it only involves whole numbers.

Aesthetic experience in constructing mathematical systems

I have discussed the place of aesthetics at the levels of uncovering pattern and constructing proof. Now I want to discuss a construction task of a different, though related, kind. This takes us into a deeper context but one which, like proof, provides opportunities for invoking aesthetic ideals. Once again it involves creativity, not *within* a system, but rather the creating *of* a system.

Beardsley (1958) in discussing *unity*, an aspect of form which was mentioned earlier by Charlton, suggests that it makes sense to speak not simply of an object's having form, but of its having more or less form. He analyses formal unity into the dual ideas of *completeness* and *coherence*. He describes completeness as a simple property, one that cannot be analysed out any further. But he does add that to attribute completeness to visual design, at least, 'is to say that it appears to require, or call upon, nothing outside itself, it has all that it needs; it is all there'. A coherent design, on the other hand, he informs us 'contains nothing that does not belong'. He gives examples of these qualities in paintings and music, and also questions whether such qualities can be spoken about in the same way in distinct art forms. Whether such qualities can, therefore, be discussed in a similar way in mathematics must at the outset seem even more dubious.

Nevertheless, what is striking is that 'completeness' is explicitly used in mathematics, and 'consistency' is used alongside it, having a meaning very close to that of 'coherence'. Both terms are most readily applied to axiomatic systems.

The axiomatic approach has already been introduced in the discussion of Kline's and Sullivan's view of mathematics. It involves constructing mathematical systems which often represent structures which are of practical concern. For example, the structure of switching circuits can be represented by Boolean Algebra. That is to say that an abstract set of symbols has been constructed which correspond to switches and their operations, so that by manipulating symbols we may make accurate predictions, indirectly, about the behaviour of switches. The choice of axioms in a given system is to some extent open, and mathematicians employ aesthetic ideals, like economy, when selecting such axioms. In this sense it would be unpleasant to have 'redundant' axioms, a situation which arises when the same theorems of the system could be deduced using fewer axioms than those chosen. Moreover, the system must at the same time be consistent. It must not allow a theorem and its negation both to be deduced from the axioms.

At this point an important difficulty arises. It has been supposed that if we wish to adopt the aesthetic attitude then in a sense this involves suspending considerations of the usefulness of mathematics. The two are mutually exclusive. However, in the contexts considered so far no obvious tension has arisen between the aesthetic aspects and the utility of mathematics. They have simply been different sides of the same coin. But a slight tension arises in the present context when we confront pedagogical and aesthetic ideals. Using the axiomatic system of the mathematical group structure as an example, Griffiths and Howson (1974) point out that it is often necessary to suspend the aesthetic ideal of economy and present students with systems that do contain redundant axioms. They write:

> ...when setting out the group axioms it is usual to demand the existence of both a right and a left neutral element, and a right and a left inverse, whereas the existence of a right neutral element and a right inverse element will suffice. In doing this we select axioms which emphasise symmetry, and simplify technical work, i.e. are pedagogically desirable; rather than those that are most economical and therefore logically desirable. (p.230)

They add that '...economy of axiomatic systems tends to be an aesthetic criterion of mathematics rather than a practical virtue of mathematical education'. What they have in mind here, as well, is that it is sometimes necessary to take a point further along the deductive chain from the axioms and base a pupil's mathematics upon such a point as though this point were an axiom when it is more correctly a theorem. Such points Griffiths and Howson call 'pedagogical axioms'; to use these pedagogical axioms is:

> ...sometimes a way of introducing pupils to work within a branch of mathematics, long before they are mature enough for a detailed proof that the axioms are satisfied. (p.230)

This point is well taken. All the same, it is interesting to note that *symmetry*, itself an aesthetic notion, is the pedagogical consideration which replaces the ideal of economy in the example given here.

Aesthetics and utility

So far we have considered three aspects of aesthetic experience in mathematics. These involve uncovering pattern; constructing proof which meet the aesthetic standards such as simplicity and economy; and constructing axiomatic systems from which results and proofs arise. I have suggested that we can at least begin to explain these aspects as being a source of aesthetic experience by making reference to certain principles found in the works of aestheticians. In this way 'aesthetics' in mathematics is intelligible and does provide us with some reasons for studying it.

But even if aesthetic experience is possible and may be sought after, it still remains to be seen how *stable* this is, as a reason for learning mathematics. We have seen that the notion of aesthetics implies a non-purposeful attitude towards mathematics, one in which enjoyment is forthcoming. Nevertheless, whilst mathematics may be a source of pure enjoyment for a *pupil*, such enjoyment may not be the ultimate purpose which an *educator* has in presenting mathematics in this light. It is easier to explain this point by analogy with the more neutral aspect of *fascination.* It is clear that the entertainment value of mathematical puzzles and games mentioned in the Cockcroft Report (see above page 51) is *sui generis* and that thus some learning of mathematics enlarges one's options in life with respect to a special kind of entertainment. Similarly, some facility in language can provide opportunities for entertaining one's self with crossword puzzles and the like. But this consideration is distinguishable from the use of mathematical games which an educator might make in order to motivate a pupil to learn mathematics for its supposed utility. Thus a pupil might be given an investigation or game, which does indeed fascinate him or her, but which the educator has chosen primarily because it embodies mathematics of some utility. So fascination in mathematics may be one perfectly self-sufficient terminus for learning mathematics or it might be used as a means for other purposes. It is in this sense that we can speak of the fascination of mathematics as not being an especially stable reason for learning it, since it may always rest upon a sensitive line between means and ends. The same is true of aesthetics in mathematics.

Sawyer, as we have seen, was prepared to define mathematics in terms of pattern. But it is very important to note that to invoke the aesthetic aspects of mathematics in this way is, for Sawyer at least, to explain only the *motivation* for developing mathematics. That is to say, that although we may agree that the aesthetic aspects of mathematics attracts mathematicians and therefore gives them a reason for studying and researching the subject, the aesthetic dimension is not for Sawyer the end of the matter. The final test in justifying mathematical research, for him, is whether or not the mathematician does serve his or her community in some way. To be sure, Sawyer uses a state of emergency to make his point. He sets out to justify a mathematical department in the developing Gold Coast during the middle of this century. He writes 'to defend mathematics in such circumstances *purely* on the grounds of its beauty is the height of heartlessness' (Sawyer, 1955 p.16). Nevertheless, in order for a mathematician to develop useful mathematics Sawyer points out that he or she must at least be interested in the subject. 'The fascination of pattern and the logical classification of pattern must have taken hold of you'. This is not unrelated to the situation in which pupils are introduced to games during mathematics lessons. From the *inside* the pupil is engaging in mathematics for its entertainment. But, as I have said, this is quite consistent with an educator's justification that the value of mathematics is its usefulness. Games for the pupil, like the aesthetic properties for the mathematician, are a source of motivation.

Sawyer's position is not an isolated case. More recently, Dreyfus and Eisenberg (1986) have written that 'one of the major goals of mathematics teaching is to lead students to appreciate the power and beauty of mathematical thought'. Although they are less explicit than Sawyer it is clear that these authors, too, are in part aiming at aesthetic appreciation in order *to facilitate mathematical thinking*. Dreyfus and Eisenberg cite both Poincaré and Papert who are impressed by the role which aesthetics plays in reaching solutions. Papert in asking for the nature of mathematical aesthetics asks how it can *drive* mathematical thinking. This clearly suggests that aesthetics, here, is not simply an end in itself, any more than appreciating the aesthetics of an intentionally persuasive speech, or appreciating the aesthetics of a marketable motor car design, is similarly an end in itself for the speaker or the designer. Of course, there need be nothing disingenuous in finding effective ways of inducing mathematical thinking, so long as we do not think that we have thereby solved the general question of why we learn mathematics.

The *non*-experienced aesthetic dimension of mathematics

Poincaré's and Papert's views, to which I alluded just now, need to be drawn out a little further since they both identify another aspect of aesthetics in mathematics. Moreover this is an aspect which is supposed to be a crucial, though perhaps under-valued, facilitator in the process of arriving at mathematical results. Earlier we saw that there is an established conceptual distinction between what is aesthetic and what is purposeful. I now want to consider an application of 'aesthetic' in mathematics in which the contrast in question is not so much between the purposeful and the non-purposeful, since this dichotomy is one connected with a *conscious* attitude. Aesthetic judgement now applies to *un*conscious thought processes and it provides a different contrast, one between the logical and the extra-logical. Such a view has its origins in the writings of the mathematician Poincaré. In his *Mathematical Creation* he writes:

> A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms *placed in a certain order*, and the order in which these elements are placed is much more important than the elements themselves. (See Newman, 1956 pp. 2042-3)

What concerned Poincaré was how the order in which the mathematical solution arises from the mathematician. He gives an autobiographical account of a problem which he solved in stages, not all of which he was conscious of:

> Among the great number of combinations blindly formed by the subliminal self, almost all are without interest and without utility; but just for that reason they are also without effect upon the esthetic sensibility. Consciousness will never know them; only certain ones are harmonious, and, consequently, at once useful and beautiful.[96] (p.2048)

Poincaré believed that those particular ideas that come to mind are those that have been selected unconsciously by an aesthetic sense. He does, however, think that such a

[96] I am assuming that Poincaré's mention of 'utility' and 'usefulness' here are to be taken as utility and usefulness *within* mathematics.

selection is a function only of distinguished mathematical minds. However, his ideas have been taken up and developed by Seymour Papert (1993). Papert claims that currently we view mathematics as a subject regulated by *logic*, and that the reason why Poincaré can identify the special aesthetic sense with the exceptional mathematician is that this sense is simply not exercised in mathematical learning. This lack of training Papert seems to think undermines Poincaré's view that the necessary aesthetic sense in mathematics is reserved for the few. He carried out some empirical work on various people that shows that the selections of manoeuvres which they make in proving a mathematical result, converge towards a particular manoeuvre which leads to a fruitful way of proving the result. His work is empirical, but it does give some reasons for evaluating the stage that his experimentees all seemed to reach, which give us some idea what might be meant by mathematical beauty. One difficult question which does remain is whether one's aesthetic sense can be educated by drawing out the aesthetic aspects of mathematics. But even if it can be educated the purpose of this, as I have already suggested, is not especially stable.

Summary and Conclusion of Part 2

It is important to draw things together to see what the foregoing analysis in this part reveals. I have shown that art and mathematics have been connected by a number of writers in two broad ways. Although some writers have not, as it were, gone all the way, amongst those who do are those who assimilate art to mathematics, and those who assimilate mathematics to art. The views of those who assimilate art to mathematics may, arguably, be set aside for the present purposes. Such views may, at their weakest, suggest only that mathematics is useful *in* art, in which case mathematics is justified once again, as it was in Part 1. This assimilation need not take us *beyond* utility. On the other hand, if art simply *is* a particular manifestation of mathematics then the rationale for mathematics education rests, at this point, upon the nature and justification of art.

But the notion 'art', itself, has at least two broad senses. It can mean *an* art or it can denote the *fine arts*. The disparate set of activities of teaching, lighting a pipe and picking pockets, for example, may all be described as arts. 'Art' here modifies a skill but does not imply that such a skill is universally valued. Thus to say that mathematics is an art can scarcely be controversial. It need only imply, as we saw in my discussion of Fitch, that mathematics is a set of potentially useful routines. To assimilate mathematics to the arts in this sense is thus a fairly weak claim and has little bearing upon the *purposes* of teaching the subject, over and above reasserting its utility. 'Fine art', on the other hand, does usually denote artefacts of universal value which it is one task of education to reveal. Thus any claim which assimilates mathematics to fine art is a very strong one to make. Nevertheless we have seen that such claims have been made. Kline and Sullivan have been my main spokesmen, but many others have, in varying degrees, suggested by their remarks that they, too, believe that mathematics is at least associated with fine art in some way. Now, since they all take it for granted that art is already of value, these claims all provide what I have called 'justification by association'. The claims have been made on the strength of certain persistent features, associated with the development or learning of mathematics, such as creativity; aesthetic properties (in particular, beauty); the expression of emotion; and self-knowledge. I have tried to show that none of these claims is satisfactory. Whilst it can be shown that mathematics does provide evidence of all of the foregoing aspects, to some degree, the fine arts are better placed to provide each of them. Moreover, since the fine arts are much more firmly rooted in human perception than mathematics, the latter, I have argued, could never be categorised in terms of the former.

The upshot so far is thus negative. But more positive results do arise. My discussion emphasises a point often made but which is important to stress. It is that aesthetics and fine art are not *coexistent.* Thus to refuse to categorise wholeheartedly mathematics as fine art, is not to deny that mathematics cannot be a source of *aesthetic* education in certain ways. Aestheticians do not readily cite mathematics as an object of their study, but those who regard aesthetics as a special kind of perception, or better still an attitude, have to admit that almost anything can be a subject for aesthetic experience. The formal properties of pattern in mathematics are thus a strong candidate for such experience. But there are other aspects of mathematics where aesthetic ideals such as elegance and economy are appropriately striven after. In addition it seems that an aesthetic sense regulates the production of mathematical results, not in intentional attentive activity but within the unconscious mind. This I have called aesthetic *non*-experience.

As the aesthetic aspects of mathematics have some clear educational implications I shall spell these out a little more. I have endorsed the claim that there are aspects of mathematics that can provide a source of aesthetic experience of a certain kind. Since such experience is a form of *enjoyment* it follows that mathematics may be pursued, in part at least, for this reason. At one level such enjoyment arises simply from exploring the ubiquitous patterns in mathematics – both following them discursively and contemplating the unity expressed in their summary formulae perhaps. At another level one may not simply marvel at the aspects which are already within mathematics. One can play a part in producing mathematical arguments which establish results already perhaps revealed in patterns. At this level aesthetic experience is gained in meeting aesthetic ideals, such as conciseness. Mathematical arguments, some of which may be formally accepted as proofs, may be presented in ways which are pleasing to follow and contemplate. Of course one need not *produce* mathematical arguments. At the first level of engagement one can still appreciate the arguments that others have produced. At yet another level one may even construct, not simply mathematical arguments, but a whole mathematical system which provides the source of mathematical argument. Such a system is typically subject to meeting aesthetics ideals of economy and thus elegance, for example. Finally, it may be possible either through the level just outlined or by other means, to increase my aesthetic sense to the extent that more sustained mathematical 'problem-solving' is facilitated *unconsciously.* There are, thus, levels of engagement in mathematics all of which, with the exception of the last, provide *direct* opportunities for enjoyment. Since we may thus enrich our lives, nourish ourselves or find entertainment in mathematics, it has in this respect educational value beyond utility.

However, what is not clear is how *accessible* this enjoyment is and thus whether it is more easily sought elsewhere on the curriculum. Even if it is fairly accessible, and writers, as we have seen, are less clear on this as the remarks from Leibeck, Poincaré, Young and Papert, for example, have shown, more might be expected of an educator than is normally supposed in order to ensure that pupils come to avail themselves of such aesthetic experience. More empirical work is needed of the kind outlined by Papert (1993) and Dreyfus & Eisenberg (1986) in order to throw light upon this question.

Part 3 - Mathematics and Mental Training

Chapter 8 – Mathematics as a form of mental training

Mathematics as a vehicle

In Part 1 I introduced the idea that mathematics provides a form of mental training. There, I was more concerned to see whether or not mental training was yet another example of the usefulness of mathematics alongside the other typical examples that have been continually set out in rationales for learning the subject. We saw briefly how Plato in antiquity and Fitch at the end of the nineteenth century had both distinguished mental training from the purely practical purposes of mathematics. We saw, too, that whether or not this distinction could be maintained depends upon how much the notion of 'usefulness' is extended. Whitehead, in suggesting that mathematics can provide a kind of mental training, was quite prepared to suggest that such training was part of the utility of mathematics. Perry, too, in his zeal, attributed almost all of the value of mathematics to its usefulness and Davis and Hersh, with a touch of irony, followed suit.

I have suggested already that we do have certain grounds for marking off the purely practical purposes of mathematics from the mental training purpose. This is because when it is said that mathematics has practical value this usually means that it is of *direct*, if not immediate, fairly specific use to whoever has learnt it. As I remarked in Part 1, counting and elementary arithmetic are amongst the clearest examples of the practical aspects of mathematics which are useful for everyone. But the direct, practical use of certain aspects of mathematics, whether immediate or not, may be contrasted with a different use of mathematics, namely, that use which is made by an *educator* to achieve a less specifically mathematical long-term purpose. It is within this more general educational purpose that theorists often place the value of mathematics as a form of mental training. Moreover, even if an individual, whose mind is trained by mathematics, can now *do* something that he or she could not do before training, this need not involve the conscious or unconscious application of mathematics by that individual. But as I hope to show, certain theories of mental training do not simply involve an individual's being able to *do* something, and this strengthens my resolve to maintain a distinction between the mental training purpose and the more straightforward practical usefulness of mathematics.

If the practical utility of mathematics is best characterised by regarding it a something of a *tool*, then a helpful characterisation of the mental training purpose may be made by regarding mathematics more as a *vehicle*. But mathematics is not only a vehicle for this particular educational purpose, it has also been a vehicle for displaying the unusual gifts or special techniques of so-called 'calculating prodigies'. Ogilvy and Anderson (1966) give a vivid account of one or two of such prodigies. In the nineteenth century they cite Zerah Colburn who at the age of nine succeeded, amongst other things, in raising the number 8 to the sixteenth power, 281,474,976,710,656. Another example they give is of Zacharias Dase, a German born in 1847, who had reckoned out the natural logarithms (7 places) of the numbers from 1 to 1,005,000. In 1850 Dase went to England to earn money by exhibitions of his talents. Such prodigies, then, provide examples of mathematics being used not primarily as a convenient tool for solving everyday problems, but more for the *entertainment* of others, as mathematics is used as a vehicle for certain individuals to exhibit their mental prowess, even if such displays like these are perhaps rare nowadays.[97]

The entertainment aspect is broad enough to include almost all of the material discussed in Part 2 under art and aesthetics.[98] What I want to show now is how mathematics considered as a vehicle is supposed to have an additional purpose, other than mere entertainment, which makes it a fit subject for mental training.[99]

---

[97] Ogilvy and Anderson (1988) add: 'When a Colburn or a Dase multiplied two large numbers together in one twentieth of the time it would take to do the arithmetic by longhand, people were impressed partly because there was *no other way* to get the answer. But, now, when a machine can do the same "problem" in a twenty-thousandth of the time, with virtually no chance of error, the human performance, while just as noteworthy as it always was, fails to make the same impression. It happens that there have been no famous calculating boys in the twentieth century. When the next one turns up, it is a safe bet that his appearance will cause little stir' (pp. 106-107). Nevertheless, it does seem that at least one *female,* who, even if she is not regarded as a 'calculating prodigy' can still cause something of a 'stir' Consider the attention which Carol Vorderman has attracted since her long-standing regular appearance on the television programme COUNTDOWN.

[98] I say 'almost all' of this material. If mathematics were a fine art in respect of its having the value which White attributes to it, namely, that it can give us a particular kind of self knowledge, then mathematics would hardly be entertainment, or at least it would not be solely this.

[99] But even if we leave aside the exceptional cases of mathematical virtuosity, it still seems to be the case that mathematics may be used as a vehicle for display in such contexts as public examinations. Mathematics is often regarded as a particularly suitable way of displaying intellect. Indeed at least one psychologist (Skemp, 1971) has claimed that mathematics is a form of intelligence. So that displays of mathematics are displays of intelligence. Of course when such tests are used with the sole purpose of assigning a *measure* of intelligence, then mathematics may be regarded as a tool.

The ambiguity of 'mental exercise'

One question that I want to raise straightaway, however, concerns the extent to which display entails improvement or development of some kind. Suppose that I show my disapproval at something. Whilst this display might bring me some satisfaction, it would be odd to suggest that in displaying disapproval I thereby improve or extend my ability to disapprove or become more disposed to disapprove on future occasions. Thus a display of disapproval is in this sense *mere* display. Sometimes, 'exercise' is used in a similar way, so that when I exercise my authority in a given situation I *exhibit* it. But my authority is not increased by such exercise. On the other hand, in displaying physical skills I am sometimes also exercising the body and may even thereby improve or extend its capability. I display my skill at lifting a heavy object and thereby may reasonably suppose that, on some occasions at least, I improve my strength as a result of such exercise.

The question now is to what extent mathematical 'exercise' is mere display, or whether such exercise typically improves or extends the mind in some way. I remarked above that a calculation prodigy uses mathematics as a vehicle to displays his or her 'mental prowess'. But the question is whether such display *improves* the mental prowess, of the kind that the prodigy is supposed to have, in a way analogous to lifting a heavy weight; or whether it is mere display. If it is mere display, then to invoke mathematics as a vehicle at this point may seem to have little bearing upon its justification for purposes other than recreation or entertainment. Of course there is a sense in which 'mathematical exercises' are set up for the sole purpose of practising routines with the reasonable expectation that mathematical knowledge will thereby be extended. In this sense mental exercises are analogous to physical exercises. But this use of 'exercise' is not the one that writers always have in mind. For example, R.K. Elliot (1975) writes:

> It seems obvious that rigorous and systematic disciplines - `Mathematics, Physics, History, and Philosophy, for example - provide great scope for the *exercise* of the powers of understanding, and are about things that matter; and it seems obvious, therefore, that at an appropriate age children should be introduced to disciplines of this kind...[100] (p.59)

Elliot appears to be invoking mathematics as a vehicle for challenging the powers of understanding. Moreover, his use of 'exercise' suggests that engagement in mathematics does indeed have an effect analogous to physical exercise, rather than the mere exercise of authority. In other words, that mathematics not only provides opportunities for *exhibiting* mental behaviour of a certain kind, but also thereby improves, develops or extends such behaviour. As I have suggested, when we speak of 'exercising our authority' we mean only that our authority is being used or appealed to. In physical contexts, unlike mental contexts, it seems to be harder to distinguish exhibition from development or improvement. When we take up a sport as exercise, this allows us to exhibit physical performance and typically, if not necessarily, to *increase* our strength or agility. But the assumption that the same occurs in the mental context is questionable. Elliot wants to justify mathematics, and other subjects, partly as a form of mental exercise. But we need to be sure that the notion of exercise can be applied to mental and physical contexts in a similar way. Another example in which exercise may be mere display is in the context of a *quiz*. Here, the quiz is a vehicle which provides opportunity for exhibiting knowledge. But it is still an open question whether one's knowledge is extended through a quiz especially if those taking part are not given the answers but only an indication of whether or not their answers are correct or not.

The foregoing discussion shows that it is important to clarify carefully the nature of any mental training justification for learning mathematics. If 'exercise' is one of the preferred ways of drawing out this special role of mathematics, and if mathematics is thus being used as a vehicle, then we must be clear that it exercises *and* improves the mind, and be assured that it is not mere display. But the view that mathematics does provide certain special opportunities has had a long history, and as we shall see it takes on a variety of forms. Although this particular rationale for mathematics education has not been emphasised in the last few decades as it was previously, it is still a view which is upheld in influential educational documents. I shall give three examples of these.

Recent statements of mental training as a justification for learning mathematics

(i) The 'Handbook of Suggestions'

The *Handbook of Suggestions* from the HMI series 'Mathematics 5-11' (1979) acknowledges that one of the purposes of teaching mathematics is that mathematics

---

[100] My italics

trains the mind. A little more is said about the mental training claim. It seems that it is 'open to abuse':

> Mathematics can provide a valuable mental training, but many other things can do this just as well; and mathematics cannot be justified on this ground alone if it deteriorates into stereotyped working to rule, with the higher functions of the mind neglected. *Mathematics can be justified as a training for the mind*, but the training also needs to serve other purposes which can be understood by the pupil at the time. There is something wrong with the teaching if the reply to 'Why are we learning this?' has to be 'You will understand later on!' [101] (pp.4-5)

But suppose that such mental training does not 'deteriorate into stereotyped working to rule, with the higher functions of the mind neglected', how and why can mathematics be then justified on the grounds that it provides mental training? The above passage from the booklet states that it *can* provide such training, but nowhere in this publication does it say *how*, and hence an answer to this question must be sought elsewhere. Moreover to satisfy the demands of this HMI document, at least, we would need to be clear about certain other notions. We would need to consider not only what is meant by 'stereotyped working to rule' and the nature of the 'higher functions of the mind', but also certain other fundamental aspects of the nature of mental training arising from the above extract. These may be summarised as follows:

(i) Training the mind is not exclusive to mathematics.[102]

(ii) The learning of mathematics *per se* is not sufficient for such training; certain pedagogical conditions are also necessary.

(iii) The effects or value of the type of mental training achieved through the learning of mathematics in the required way may not be something that is immediately obvious to the pupil, and mathematics must also have some instantly recognisable worth of some kind.

(iv) Implicit in this passage is a distinction between two senses of training which may take on the form of a dilemma: there is training as *mere repetition or drill,* which is thought to be somehow wrong, and training which requires (or promotes) *intelligence*.

---

[101] My italics

[102] This much we have seen is true of writers like R.K. Elliot who also includes physics, history, and philosophy, alongside mathematics, as suitable for exercising the mental powers.

'Mere repetition or drill' ***versus*** **'intelligence'**

The ambivalence concerning the nature of mental training in the document is not simply a muddle. We shall see that the whole ideal of mental training is characterised by different aspects of mathematics, which are supposed to be significant in such training. One way that I want to draw out these aspects will be in terms of a duality between what I shall call the *contemplative* and the *procedural* aspects of the subject. The reference to 'mere repetition or drill' in the HMI document, I hope to show, can be aligned somewhat towards the procedural aspects of mathematics, whilst the reference to 'intelligence' might be considered, in part, to link more with the contemplative aspects. I say 'in part' because intelligence may be displayed in practice as well as in reflection. The correspondence is not by any means perfect but it will draw out what I think is one of the perennial dualities implicit in mental training aims in mathematical education, from ancient times to the present day. I shall be discussing this duality in detail using in particular Plato and Descartes as important theorists who draw out the reflective and procedural aspects of mathematics for mental training ends.

'You will understand later on!'

I remarked earlier that one of my reasons for supporting the view that mental training is not mere utility was that the mental training aim is sometimes a long-term aim. The HMI document, too, seems to assume that a long-term aim is at stake in mental training, and this is seen as presenting a particular kind of conflict for the pupil. I shall discuss another aspect of mathematics that is supposed to provide a relatively short-term aim of mental training since various theorists have pointed out that there is something rather simple, and immediate about mathematical *evidence* that makes it an ideal source of mental training. This view can be traced back at least as far as St. Augustine, and is drawn out particularly well in the nineteenth century by Thomas Tate whose account amongst other more recent related ones will be discussed below.

(ii) The Cockcroft Report

The HMI booklet does not, however, present an isolated case for mental training in recent years. *Mathematics counts* (Cockroft et al, 1982), also suggests that mathematics can be justified as a form of mental training and this is more explicitly given as 'powers of logical thinking, accuracy and spatial awareness' (p.2). The report goes on to reiterate the cautionary remarks of the earlier document noting that it '...depends upon the way in which mathematics is taught' and also that '...the study of a number of other subjects can develop these powers as well'. Whether or not 'accuracy' in general, rather than simply mathematical accuracy, can be improved by learning mathematics is questionable. The same seems to be true of 'spatial awareness' though this is often an assumption that teachers make from time to time in teaching 'shape and space'. But the belief that 'logical thinking' is developed through mathematics has been an established view. Mathematics is, in a sense, a logical discipline, though sometimes there seems to be confusion regarding the question of whether a logical mind is *presupposed* for learning mathematics, or whether a logical mind is *the result* of learning it. This aspect of mental training will be addressed in my discussion of what I am calling the 'procedural aspects' of mathematics.

(iii) The National Curriculum in France

It is interesting to note that the current National Curriculum in France also embodies the belief that one of the purposes of mathematics education is to provide something more than utility. Writing about the provision for the country's lower secondary aged pupils Howson (1991) writes:

> In a manner which hints at 'old-fashioned' faculty psychology but which, nevertheless, must still have some justification, it is pointed out that mathematics education should assist the intellectual development of the pupil through:
>
> developing reasoning powers: observation, analysis, deductive thought;
>
> stimulating the imagination;
>
> promoting the habit of clear expression, both written and oral;
>
> stressing the qualities of proceeding methodically and with care. (p.74)

I will not say much about these supposed purposes of mathematics. Some of them, 'developing reasoning powers', 'deductive thought' and 'stimulating the imagination', all arise in my discussion of contemplative and procedural aspects. What I have already said about the 'accuracy' purpose of the Cockcroft report, I think, can be said equally of certain aims of the French National Curriculum such as promotion of 'clear expression' and 'care'. The latter notion of 'care' however might very well be harnessed to the idea of reasoning and logic. As it stands it is too vague to elaborate upon.

The main point of outlining the latter two sources is to show that the belief in such extra-*utility* reasons for learning mathematics is still very much alive today. But since these beliefs are expressed in rather pithy expressions, the need to examine such claims that mathematics can train the mind is essential if it is to provide a rigorous means of justifying the learning of mathematics. So far, I have spoken of 'mental training' in this part of my account, as I did in the first part, as though it were a clear notion. This is not the case. Indeed, writers do not always use the expression 'mental training'. In some cases writers seem to be referring to something like 'character training' where certain dispositions are to be acquired. At other times it is more like an ability which is desired. Sometimes it is rather more like a special form of knowledge such as 'wisdom', 'sagacity' or 'intelligence'. I shall continue to use the expression 'mental training', as a convenient way of referring to all of these, but the varieties of mental training will be drawn out in the following discussion.

The concept of mental training

If we consider firstly the general concept of training we see at once that it is a broad notion. We not only speak of 'toilet training' for infants, 'house training' for animals, but also of training plants to grow through a trellis for example, and of training the hair on our head to grow in a certain fashion. Sportsmen and musicians train, too, and so do teachers, scientists and scholars. The first question to answer is whether or not these examples of training all have something in common that can help us to determine the precise nature of *mental* training. Certainly the outcome of the training in all of these cases is that some *intentional change*, on behalf of the trainer, has taken place in whoever or whatever it is that is being trained. But although this might be set out as a necessary condition of training it is not sufficient. Intentional changes might follow as a direct result of chemicals, or medication of some kind, so that the *method* used for making intentional changes plays a criterial role in all except perhaps certain borderline cases. What is usually required is intentional change as a direct result of the efforts of the trainer.

More can be said about methods of training. Training involves some measure of *repetition* and in many cases application to different situations. We do not usually train something or somebody by a single action or even a few attempts. Training a plant seems to be a borderline case in this respect if all that is required of the trainer is his or her weaving it through a trellis. Nevertheless, even here, as elsewhere the trainer must always be aware of the particular kind of change required and will be prepared to *intervene* if the training is not succeeding. So training seems to imply that during some period of time there is always the possibility that the required change will go off course and repeated intervention is expected. Not only is training usually restricted to a certain context, training to be an actor, for example, is linked to the context of entertainment, there is also a sense in which training has *relatively* specific ends. This is certainly true of simple cases of training, like drill. But we have to be careful here since teacher training, for example, is far from being a closed and precise activity. Nevertheless, to the extent that we can speak of teacher *training* rather than teacher *education* certain fairly clearly specified goals are implicit and also a measure of repetition is required. But as training is applied to more complex cases, like teaching, the importance of application to a variety of situations both familiar and novel becomes important, even if those situations are themselves set within fairly specified bounds.

The intentional change that a trainer brings about, through a certain method, does have its limits. We can train many people in life saving, some people in heart surgery and no one in fortune telling. Something or somebody can only be trained in that for which they already have a *capacity* and maybe even a *tendency* or *inclination* of some kind. The hair and the plant that we train already grow, but it is the direction of the growth that is important to the trainer. To leave something untrained is to let it follow its own tendencies which may appear to be random or capricious, yet which often might be those which are linked to other important natural functions which have purposes intrinsic to them. Thus to train something or someone is often to set out to do something against an inclination to do otherwise, and a trainer will usually have a purpose for training, or is, in principle at least, always accountable in this respect. But often the training is voluntary and the trainee is aware that the trainer will be intending to make changes of some kind. A potential lifesaver already has a capacity to save lives. He or she can swim well, can keep a cool head, has a degree of courage and is prepared to take some risks. The trainer builds upon these prerequisites, extends them and introduces others necessary for the desired end. The trainee surgeon already has some knowledge that will need to be extended. More importantly, he or she will be trained to be able to apply that knowledge to a range of novel situations of which the trainer has usually had experience and is in a position to provide instruction, demonstration, exercise, praise, correction and warning.

We could, perhaps, generalise this idea of *tendency* to all cases of training. For example we might want to say that the untrained footballer who once had a natural tendency to kick a ball has, after training, a tendency to kick the ball more efficiently, accurately and purposefully. Similarly, we might say that the would-be scholar who had a natural tendency to research tends to be, after training, scrupulous and unprejudiced in his or her discussions and arguments, or much more so than hitherto. But we should not stretch this notion of tendency too far. Training involves learning[103] and often knowledge and understanding. Yet to distinguish training from mere teaching it is helpful to consider it as also involving *modified* tendencies, or inclinations. In short, training sets out to form fairly settled dispositions in its trainees.

By considering the degrees of knowledge and understanding involved in training of various kinds, and the extent to which application to novel situations is important, there appears to be a training continuum. At one end training constitutes a drill, where one is trained to carry out some relatively simple task as second nature. At the other end some settled disposition is expected as part of the training and a considerable degree of understanding is required too. A trained surgeon carries out skilled procedures, which might otherwise have been rules of thumb. He no longer has to rely on sheer luck. But the procedures of the trained surgeon whilst involving repetition and even some drill, are complex and cannot be solely a matter of drill; some understanding and deliberation is necessary in such cases. The same is true of teacher training. A teacher is not trained merely by drill. Knowledge and understanding are involved as well, though these are not sufficient. The training will involve, as it does for the surgeon, applying knowledge to situations under the guidance of one who is experienced in such situations.

On this account of the concept of training, '*mental* training' involves a desirable, intentional, mental change taking place in individuals who already have some capacity for such change, by the continual intervention of a trainer. Some kind of settled mental disposition would be required too. Such training might range from mere drill, on the one hand, to a training which involves knowledge and understanding, applied to a range of situations, on the other. This much can be inferred from the HMI document above.

---

[103] This cannot be true of plants and hair since they are not agents and do not learn to do anything. Indeed for more passive recipients of training the best we can say is that training implies that some change has taken place usually by some repetition such that a certain *tendency* has been overridden and replaced by another supposed better one.

But it still leaves unanswered the question of *how* mathematics can train the mind, and even if it can what *kind* of specific mental change is forthcoming from this process.

In learning mathematics, as in any other subject, we may acquire a capacity which is either dangerous or useless.[104] So if mental training is worth having we shall still need to know why. The obvious answer seems to be that such training will be of use to us, but this means that the distinction which I have been trying to uphold, between learning mathematics for utility and non-utility reasons begins to collapse. On the other hand if mental training is some kind of modified tendency, settled disposition or valued character trait, such training might contribute to our personal well-being in a way that utility does not, of if it does, only indirectly. *Whatever the outcome of such training, however, we must be sure that it is not simply the ability to do mathematics or even the tendency or disposition to do or want to do mathematics, otherwise this supposed justification for learning mathematics will be circular.*

---

[104] For a discussion of this point see Passmore (1980)

Chapter 9 – Self mastery

Meditation – Training the restless mind to become calm – a test case

A clear example of mental training, which satisfies at least some of the criteria for mental training outlined in the previous section, is to be found in *meditation.* This is an activity undertaken not only in religious contexts, where it is a means of reaching a mental state of devotion, but also more recently in the secular context of therapy where it is advocated as a form of stress relief. The stress in question is mental stress, which, it is supposed, can be relieved by the particular process of mediation. The need for meditation arises for many of its advocates from the undesirable natural tendency of the untrained mind to flit from one object to another. Thus meditation is a form of training which is supposed to submit the mind to conscious *control*. It is believed that such control will promote mental health and well-being.

This method of training is easy to describe in general terms. Unlike the training discussed above, in meditation one is usually one's own trainer. According to many accounts, this training simply involves trying to focus all of one's attention upon a fixed object of one's choice. Whenever the mind wanders off this object, which it frequently does in the early stages at least, the attention is to be brought back gently to the object of focus again. With practice the ability to focus is gradually prolonged and the benefits, some of which are supposed to be immediate, are increased. But it is not essential for this training to contemplate a fixed object. A more discursive procedure is also suitable using for example the first ten counting numbers. So that slowly counting from one to ten, repeatedly, attending to nothing else but each number in turn is *one* method of training the mind. In an extremely minimal sense, then, we can say that some mathematics may be used in meditation for mental training. We can agree, too, that other subjects will do just as well, simply because the nature of the subject matter is of little importance. But it is not even necessary to carry out a cyclic activity of this kind since, according to its practitioners, there are other suitable *procedures*. Meditation may be carried out simply by *contemplating* any object of one's choice. So that within meditation we find that the two aspects which I mentioned earlier, the contemplative and the procedural, can both be at work in this simple but clear test case.

Meditation, then, provides us with one extremely simple model of mental training where mathematics *can* be used to train the mind to become 'still'. The end state is a state of mental change and it is a valuable one too, since it is one of personal well-being. But the particular use of mathematics which I have given is extremely minimal since the only aspect which is operative is sequential order. Two important aspects of mathematics are its concern with quantity, and deductive reasoning. In meditation no use is made of either of these aspects. In fact any ordered set of repeated words would be just as effective. Moreover the process is one of *drill*, to the extent that no additional knowledge or understanding, over and above the training method and its expected outcome, is essential. In particular one does not need to study mathematics for this kind of training. So the learning of mathematics could hardly be justified for this purpose.

But the case of meditation is helpful. It shows that the outcome of mental training need not involve the mind's being able to *do* something. Certainly one has to practise the focusing technique, and this takes time and patience, but the outcome is not simply a skill in focusing. The fruits of meditation *presuppose* success in focusing the mind so that meditation trains the mind to *be* still or calm. It does not give the meditator some specific ability of any kind, though of course it may remove certain obstacles preventing other abilities from being formed. For example it may provide certain conditions for aesthetic experience, since as we have seen this involves a kind of detached contemplative attention. What I want to emphasise, is that since meditation clearly involves control, it forms part of what we may call 'self-mastery'. I have chosen this particular expression since it is the one adopted by at least one important theorist who did believe that mathematics played a central role in a particular kind of mental training. Thus we may broaden this simple test case of meditation into something more substantial.

I should point out that I am not concerned with any function that meditation might have in special contexts like, for example, religious ritual. I have been considering the most simple of cases where an overactive mind needs to be calmed by some mechanistic drill. Nor do I want to suggest that meditation is a way of combating the dominance of certain *desires*. I want to remain neutral about the kind of thing that is causing unrest in the individual. Nothing in my account should suggest that the meditator wants to reach a calm state for any other reason than that the current state is uncomfortable and prevents him or her from flourishing in some way. My account, then, entails no further moral considerations of this discomfort. Indeed the mental turbulence or unrest may just as well be the result of considerable time spent in altruistic actions, abstract speculations or avaricious planning.

What we now must seek are fairly clearly-cut examples of mental training, like meditation, where the outcome of the training is characterised as some kind of extra-mathematical human well-being, but where mathematics is supposed to play an essential role. These I believe can certainly be found in the works of Plato and to some extent in Descartes. Like most rationalist philosophers these men took mathematics as either the starting point, or the essence, of knowledge, and so it is the question of how the acquisition of knowledge may be regarded as training, in a non-trivial and non-circular sense, which needs to be drawn out. Even if Plato and Descartes do not always explicitly refer to mathematics as a form of 'training' then certain of their commentators certainly do. In this respect these two philosophers in their different ways will form the backbone of my enquiry.

Plato and self-mastery

Education as a tranquilliser.

In his examination of the nature of the just state, in *The Republic*, Plato draws his attention away from the state and directs it towards the soul of the individual. He makes an analysis of the soul expecting to find there, features analogous to those of the state. Improvement amongst these features he hopes will thus lead to the just state. The soul, he argues, has different elements. It has both an irrational appetitive part and also a reflective reasoning part, which are potentially in conflict and need to be kept in balance. In order to achieve order amongst them a measure of *self-mastery* is required. He explains what he means by the expression *self-mastery* in the following way:

> 'What the expression is intended to mean, I think, is that there is a better and a worse element in the personality of each individual, and that when the naturally better element controls the worse then the man is said to be "master of himself", as a term of praise. But when (as a result of bad upbringing or bad company) the smaller forces of one's better element are overpowered by the numerical superiority of one's worse, the one is adversely criticized and said not to be master of oneself and to be in a state of indiscipline' (Plato, 1955 edn p.201).

Self-mastery,[105] then, occurs when 'the better part' of the personality 'rules the worse'. So here Plato has described a situation similar to that which the stress therapists is

---

[105] Elsewhere, too, Plato draws a similar picture of the soul:
'...within each of us there are two sorts of ruling or guiding principle that we follow. One is an innate desire for pleasure, the other an acquired judgement that aims at what is best. Sometimes these internal

confronted with before meditation. In both cases there is some kind of over-active state which has somehow to be stilled, or a tension which needs to be resolved. Of course there is a difference between the two. The stress from which an individual suffers need only be temporary and not a settled part of his or her nature. All the same, meditation is seen as providing some kind of control. Moreover, the therapist need not speak of wrong and right *forces*, only, more neutrally, of comfortable and uncomfortable mental states. For Plato, however, the initial condition is more specifically a moral case of wrong and right forces. The wrong forces are the passions that need to be kept in check. He draws a picture in which the mind consists of two elements, reason and passion, which are in potential conflict. To these two elements he adds a further irreducible one of 'spirit', which is exemplified by the part of the individual that becomes angry, or indignant, when one has given way to irrational appetite. But whilst it is a third element, Plato points out that it fights 'on the side of reason'. Goodness arises from the harmony among these three elements.

We have seen that the therapist typically recommends a repetitive method of stilling the mind. For Plato, on the other hand, an *education* programme is required in which a harmony among the three elements of the mind is achieved by:

> '...a combination of intellectual and physical training, which tunes up the reason by a training in rational argument and higher studies, and tones down and soothes the element of "spirit" by harmony and rhythm'. (Plato, 1955 edn p. 219)

So the initial conditions of the individual before training in 'rational argument', and the individual before meditation, are strikingly similar. There is also a similarity in outcome in both cases. The fruits of meditation such as tranquillity and peace of mind, are also gained by Plato's training in self-mastery, as Charles Taylor (1989) clearly points out in his discussion of Plato and self-mastery:

---

guides are in accord, sometimes at variance; now one gains the *mastery*, now the other. And when judgement guides us rationally toward what is best, and has the *mastery*, that mastery is called temperance, but when desire drags us irrationally toward pleasure, and has come to rule within us, the name given to given to that rule is wantonness... (See Phaedrus 237d – 238 in Hamilton and Huntingdon, 1961 my italics)

> Besides being at one with himself, the person ruled by reason also enjoys calm, while the desiring person is constantly agitated and unquiet, constantly pulled this way and that by his cravings...The mastery of self through reason brings with it these three fruits: unity with oneself, calm and collected self-possession. (p.116)

It is worth pointing out that Plato's account of self-mastery was set out in opposition to at least two rival dominant outlooks. One of these was a *warrior morality*, where, as Taylor describes it:

> ...what is valued is strength, courage, and the ability to conceive and execute great deeds, and where life is aimed at fame and glory, and the immortality one enjoys when one's name lives for ever on men's lips. The higher moral condition here is where one is filled with a surge of energy, an access of strength and courage - e.g., on the battlefield - and is able to sweep all before one. It is not only different from but quite incompatible with the reflective and self-collected stance of rational contemplation (p.117).

The other, related, rival view to Plato's was one which I touched on earlier. It is, as Taylor puts it, a view which 'exalts a state of manic inspiration in which poets create'.

So there are clear similarities between the initial states of the individual who is in need of meditation (see above pp 82-83) and the individual who Plato believes is in need of education. There are also similarities in the training outcomes. The particular difference between them is the *kind* of training involved. We have seen what kind of training the therapist recommends; Plato's goal is secured by different means. It is secured by ensuring that the individual is ruled by reason, and since mathematics plays a substantial role in this, we have one of the first accounts of how mathematics may train the mind.

Education as a stimulant – a subtle shift.

What I want to be able to show is how Plato's higher educational programme, with mathematics at its core, is supposed to provide a certain kind of peace of mind. Yet it must be pointed out that the nature of the struggle between the passions, spirit and reason, seems to be transformed by Plato during *The Republic*, in a rather special way before higher education enters the picture. We expect education to be introduced as a remedy for reconciling internal conflict, given the picture of the individual's initial plight and the rival moral outlooks which Plato set out to replace. What does finally emerge, however, is a programme which addresses a particular appetite, namely the passion for *knowledge*. Indeed those who are unfit to train as philosophers are precisely those who lack a thirst for truth. Far from remedying the dominance of passion, education seems to *presuppose* it.[106] This transformation allows the struggle between appetite and reason to be displaced by something milder, namely, a contrast between perception and knowledge, since to have a passion for knowledge is to transcend mere sense perception. The emphasis is now placed on the propensity for most individuals to use their *senses*, rather than pure intellect, not only as a first resort but also typically as the sole means in seeking 'knowledge'.[107] There are of course connections between the senses and irrational appetite, but the upshot of this shift of emphasis is that education is not now so plausibly characterised as a calming agent, but as the means of stirring the individual into action. The passion for knowledge it seems needs to be aroused and maintained. The picture of the initial condition that we now have, is of an individual who is not necessarily struggling with his passions, but who may be *complacent*, believing that what is to be perceived is all there is. So, paradoxically and more importantly, unlike meditation,[108] education will be something of a *stimulant* rather than a tranquilliser.

The shift is effected smoothly by the particular example of *beauty* that Plato uses in drawing out the qualities of the philosopher. Plato has used a cunning example here, since the quest for beauty appeals both to the appetite, on the one hand, and to reflective reason on the other. Plato begins by allowing Socrates to show that his interlocutor's all-consuming passion for beauty is all of a piece with the thoroughgoing quest of the philosopher. Socrates remarks to Glaucon:

> '...anyone as susceptible as you should surely remember that those of your amorous temperament are always getting bitten with a passion for boys in the bloom of youth, and think they all deserve attention and affection' (Plato, 1955 edn p.267).

Far from suggesting that this passionate response needs to be kept in check as it may conflict with other internal forces, Plato contrasts it with the more dilettante

---

[106] A analogous position can be drawn in religion. We might say that religion is a remedy for those whose outlook is materialistic and at the mercy of the appetite. Yet evangelists are often construed as the passionate ones who are preaching to those who are complacent.

[107] I have put 'knowledge' here in inverted commas because knowledge for Plato could not be achieved from the senses.

[108] It is perhaps not wholly correct to construe meditation merely as a calming activity. Experienced meditators would surely suggest that the fruits of meditation, including as they do such goods as self

characteristics of ordinary people. He wants to bring out the point that one who really loves something loves *all* of it. In this case he is suggesting that Glaucon loves all examples of beautiful youths. The philosopher, too, must love everything; he must not be fussy. Indeed far from keeping the passions in check, the passion for loving *knowledge* must have no bounds. It is not, of course, the case that Glaucon has the wherewithal of a philosopher simply because he has such a passion for all beautiful objects of a certain kind. But it is the intensity of this passion which, when extended further, distinguishes philosophers from non-philosophers. As Socrates explains:

> 'Those who love looking and listening are delighted by beautiful sounds and colours and shapes, and the works of art which make use of them, but their minds are incapable of seeing and delighting in the essential nature of beauty itself.' (Plato, 1955 edn p.269)

What is important here is that the incapability which most people have for loving *all* beauty, which includes the pursuit of the essential nature of beauty, is not rectified by calming the passions. It involves an awakening. This is clear from the way Plato allows Socrates to characterise those who fall short in their all-consuming desire for beauty:

> 'Then what about the man who recognizes the existence of beautiful things, but does not believe in beauty itself, and is incapable in following anyone who wants to lead him to a knowledge of it? Is he awake, or merely dreaming? Look; isn't dreaming simply the confusion between a resemblance and the reality which it resembles, whether the dreamer be asleep or awake? (Plato, 1955 edn p.270)

From here onwards it seems Plato is not concerned with an excess of passions but rather with a *lack* of the passion to know the true nature of things.

For Plato no *knowledge* could be apprehended through the senses. The most that one was able to achieve in this way was mere belief or opinion. Knowledge, for Plato, was apprehension of reality; it was knowledge of what *is* rather than what is *becoming*. The 'world' which sense experience reveals is impermanent, one which is in a state of flux. Plato believed that there was another 'world'- the world of Forms - which was not subject to change, and hence it was only this that could be *known*. But sense experience could not reveal such a world; it was apprehended through the intellect. But since most people tended to rely on their senses, to apprehend knowledge of reality required an

---

possession, do allow a more contemplative state to take place in the individual, allowing a more philosophic nature to come through. Indeed meditation is often viewed as a means to self-knowledge.

enormous shift of some kind. It was Plato's belief that mathematics was the kind of discipline that could effect this shift. He shows by his famous analogies of the Sun and The Cave that the acquisition of knowledge did not involve simply putting knowledge into the individual's mind. It required the individual to make some kind of change:

> '...the organ by which he learns is like an eye which cannot be turned from darkness to light unless the whole body is turned; in the same way the mind as a whole must be turned away from the world of change until its eye can bear to look straight at reality, and at the brightest of all realities which is what we call the good' (p322).

Self-mastery, then, is achieved through education, for Plato, if the individual apprehends the Forms, in particular the Form of the Good. Moreover, the *ultimate* state is characterised by a degree of peace of mind. However, this does not show that education is simply a kind of tranquilliser. Indeed, for Plato it was more like a stimulant. The individual needed, to use his metaphor, to 'turn' him or herself in the right direction. Plato believed this 'movement' could be achieved, in some individuals at least, cognitively through reason, and more especially, that mathematics had a central role in this task. In *The Republic* this is made explicit. Mathematics was supposed to be a fit subject for training the mind to apprehend the Forms in two ways. [109] Firstly, because the study of mathematics allows the mind to enter a state between belief and knowledge, and is thus a vital stepping stone in the ascent to knowledge, and secondly because, what we would call today, 'mathematical propositions' seemed to Plato to be *puzzling*, or more strictly paradoxical. How these two features provide mental training needs careful examination.

Mathematics provides paradoxes which 'forces the mind into a quandary in which it must stir itself to think'

For Plato, anything, which was only relative to some person or some thing, was not knowledge. In this respect knowledge of *quantity*, raised severe difficulties for Plato as the following extract illustrates:

---

[109] Plato does not appear to use the either the expression 'mental training' or 'training of the mind' though at least one of his commentators does. In the Penguin translation for example Desmond Lee writes of Plato's Education of the Philosopher ' ...the main stress throughout is on the *training of the mind*...and mathematics is to be studied without any immediate practical or scientific aim in view'.

> 'And what about the many things which are double something else? If they are double one thing can't they be equally well regarded as half something else?' 'Yes.' 'And things which we say are large or small, light or heavy, may equally well be given the opposite epithet.' 'Yes, they may all be given both.' 'Then can we say that any of these many things *is,* any more than it *is not*, what anybody says it is?' (pp. 274-5)

Plato thought that there were various states of mind, determined by corresponding specific objects, and that these states of mind and their objects formed a *hierarchy*. An important role of higher education was to enable one to ascend this hierarchy. Since the highest state of mind was that which apprehended the objects of reality, complacency is the result of either taking the wrong objects as those that constitute reality, or merely inferring reality from a state of non-reality. The goal of education was an apprehension of, and love of, the Form of the Good, that which reveals the reason why things are how they are, and that which motivates our moral actions. More specifically, the complacent state of mind is one that does not distinguish between appearance and reality. Plato would say that such a mind either takes what is only *becoming* for what *is*, or it takes what both *is and is not* for what is. The reason for this is that the untrained mind is confined to relying at best on sense experience. This state of mind is for Plato at worst either illusion or ignorance and at best only belief. Neither are states of *knowing*. Knowledge is the state of mind that he seeks. So Plato's theory of mental training involves changing one's state of mind from a state of unknowing complacency to a state of *knowing*. Moreover, since knowing was, for Plato, intimately connected with goodness, his theory was of special importance.

As I have suggested earlier, Plato had a much more tightened up notion of knowledge than we have today. The contemporary notion of knowledge is either propositional or procedural, knowing that X is the case or knowing how to do Y. For Plato it was what is sometimes called *knowledge by direct acquaintance*. It was rather like the knowledge one has when one knows a friend or a place. That is to say, it involved more than simply knowing facts *about* a friend or a place. It involved a *faculty* rather like an internal form of sight, and just as sight can see objects, the faculty of knowing knew objects of some kind. Moreover, this internal vision was gained not from being given a special kind of sight, but by being *turned* in the right direction. Plato thought that mathematics provided special kinds of situations which were especially thought-provoking. He was impressed by such facts that, since six, say, is simultaneously twice three and *also half* of twelve, it is both a double *and* a half and as such cannot be a source of *knowledge*.

Plato's point here is that knowledge can only be something which *is,* and since six is and *is not* a double it cannot be a source of knowledge. States of affairs in mathematics such as this reveal to us that our understanding of what it is to be six, a half or a double are incomplete, and hence that we must continue our quest to find the nature of *all* things and not be satisfied with the testimony of the senses.

Plato believed that reflection upon paradoxes of this kind were of considerable value since they shook us out of our complacency. However, much of what we perceive is, it seemed to Plato, paradox-free in this respect, and so it does not provoke thought of this kind. We are, therefore, kept in a state of acceptance; we take it for granted that what we perceive is how things *really* are, and we do not seek *knowledge* in Plato's terms. Thus he writes:

> 'By perceptions that don't call for thought I mean those that don't simultaneously issue in a contrary perception; those that do call for thought are those that do so issue in the sense that in them sensation is ambiguous between two contraries, irrespective of distance. But you will understand more clearly if I put it as follows. Here, we say, are three fingers, the middle, third, and the little one' 'Yes'. 'And you can assume you've got what I call a close view of them. But there's a further point I'd like you to consider.' 'What is it?' 'Each of them looks as much a finger as any other, and it makes no difference whether it's white or black, fat or thin, and so on. There is nothing here to force the mind of the ordinary man to ask further questions or to think what a finger is; for at no stage has sight presented the finger to it as being also the opposite of a finger.' 'No it hasn't.' 'So there's nothing in this sort of perception likely to call for or stimulate thought.' (Plato, 1955 edn p.329)

However, since Plato thought that there is a reality which can be known beyond the objects of perception, an awareness of this reality is made possible by being confronted with these paradoxes and by reflecting upon them. Since also, mathematics provides a wealth of such paradoxes of quantity, these are *instrumental* in our quest for knowledge. Plato believed that mathematics *in this respect* was an important form of mental training. He acknowledged the utility of mathematics for some stages of education, but stressed its role as a form of mental training. He refers to another paradox concerning unity in the following passage:

> 'If our perception of the unit, by sight or any other sense, is quite unambiguous, then it does not draw the mind towards reality any more than did our perception of a finger. But if it is always combined with the perception of its opposite, and seems to involve plurality as much as unity, then it calls for the exercise of judgement and forces the mind into a quandary in which it must stir itself to think, and ask what unity in itself is; and if that is so, the study of the unit is among those that lead the mind on and turn it to the vision of reality'. 'Well, the perception of unity by sight most certainly has this characteristic; for we see the same thing both as a unit and as a plurality'. (p.331)

The point here is that since every single object can be divided, it seems that everything is *simultaneously* both one and many and once again reflection on quantity shakes our complacency, so that he concludes that the study of arithmetic,

> ...draws the mind upwards and forces it to argue about the numbers in themselves, and will not be put off by attempts to confine the argument to collections of visible or tangible objects.(p.332)

Mathematics lies between belief and knowledge

As I pointed out in the Interlude, Plato inherited from Pythagoras the view that the rationale of the universe was not some kind of material substance. But whereas Pythagoras treated numbers as the non-material ultimate explanation, Plato went further than this and introduced the Forms. This meant that numbers had a slightly lower status. Nevertheless, mathematics was still important as a way of reaching the Forms since it held a unique place in his order of things, between belief and knowing. The best one can learn about the sensible world, as I have remarked, is opinion or belief. The state of knowledge, on the other hand, is only achieved by apprehending the Forms. However, mathematics occupies an intermediate position between these two poles since it is concerned with reasoning. It lacks the intelligible status of the Forms because the mathematician still makes some appeal to the sensible world in deducing results. In doing this he is, Plato argues, in an analogous position to someone who deduces that there is, say, a physical object on the strength of a shadow or reflection of it. What distinguishes the mathematician and the higher status of the philosopher is the status of the former's starting points. The mathematician, it seems, takes the truth of his sensible geometric figure, or set of physical objects in the case of assertions about number, as given. He then proceeds to deduce results on this assumption. All that is possible on this approach is consistency. But the philosopher establishes the truth of his starting points before deducing results from them. In this way mathematics lies on the continuum between belief and knowledge. It provides more than belief but less than knowledge and thus acts as a halfway house on the way to apprehending truth.

The influence of Plato's paradox theory

It might be supposed that the paradoxes which Plato thought lay beneath relative terms, and in particular quantity, were of little significance to subsequent thinkers. But this does not seem to be true. Even if different thinkers came to different conclusions from Plato, they begin from the same point, and this shows at least that reflecting upon puzzling aspects of quantity still provides a source of philosophical enquiry. In the medieval world, for example, St. Augustine (see Howie 1969a), in a strikingly similar way to Plato, dwells upon the fact that since unity cannot be identified with anything in the sensible world, the idea of it must have its sources elsewhere:

> When I seek unity in my physical environment and am sure that I am not finding it, then surely I know what I am looking for and not finding. I must know that I cannot find it there or rather that it does not exist there at all. Thus, since I know that no physical object is a unity, I know what unity is; if I did not, I could not add up the many different parts of an object. Wherever I have learned about unity, it is not through the senses that I have found it. Through the senses I have known only physical objects, which, as we have demonstrated, are not truly and simply one. (pp. 247-8)

On the other hand, in the eighteenth century Berkeley (1962 edn) remained unconvinced by the argument that since everything is made of many parts, unity cannot be perceived. For him, the fact that an object may be seen to embody a variety of different numbers, including unity, is sufficient to show that 'number is entirely the creature of the mind':

> ...the same extension is one, or three, or thirty-six, according as the mind considers it with reference to a yard, a foot, or an inch. Number is so visibly relative, and dependent on men's understanding, that it is strange to think how anyone should give it an absolute existence without the mind. We say one book, one page, one line, &c.; all these are equally units, though some contain several of the others. And in each instance, it is plain, the unit relates to some particular combination of ideas arbitrarily put together by the mind. (p.70)

During the nineteenth century Frege (1980 edn), taking a similar starting point to Berkeley by agreeing that we can attribute many different numbers to the same object, including unity, wrote:

> While looking at one and the same external phenomenon, I can say with equal truth both "It is a copse" and "It is five trees", or both "Here are four companies" and "Here are 500 men" (p.59).

But he uses this fact not only to argue against the Platonic/Augustinian view that numbers must be 'external things'. He disagrees with Berkeley's view, too, that numbers are 'something subjective'. He argues that numbers are a property of (non-subjective) *concepts.*

More recently the apparent paradox is raised again by Geach (1957) to reach extra-mathematical conclusions, namely, the refutation of 'abstractionism' as a theory of concept acquisition. Speaking specifically of arithmetical concepts he writes:

> What number I find may vary, without my observations' varying, because I am considering a different kind of things; the same auditory experience may give me the number 2 if what I have in mind is *heroic couplets*, 4 if it is *lines of verse*, 40 if it is *syllables*, 25 if it is *words*; if I have no special kind of thing in mind, no number will suggest itself to me at all. (p.28)

This raises the question: what number could possibly be abstracted from a poem, since the number concerned depends upon the aspect of the poem that is being attended to.

As a matter of fact, then, Plato seems to be right in the sense that reflecting on this puzzle has stimulated thought in several ways, even if it does not show that the mind has necessarily turned towards the Platonic Forms. What it does show is that reflection upon some fairly simple examples of what is said in mathematics, stimulates philosophical activity. This is an important point. Plato does not stress the importance of the procedures and routines of mathematics though clearly they must play some part. It is reflection upon the nature of mathematical thinking which is his major contribution. In short mathematics is a vehicle for provoking philosophical thought and it is the philosophical mind which exemplifies self-mastery.

Nevertheless it must be said that a rationale for the systematic study of mathematics, based upon the value of reflecting upon some of the kinds of mathematical paradoxes, remains unconvincing. It is only if we introduce the metaphysics of the Forms that the systematic study of mathematics can be characterised as a point on a mental journey of some kind. But this metaphysical picture is certainly not central to the modern mind and was subjected to critical scrutiny by Plato, himself, and rejected by Aristotle after him. But before concluding the discussion of Plato, it is worth considering one contemporary example that has arisen from a school mathematics lesson which if not directly influenced by Plato certainly bears some of his thumbprints.

Mathematics as a 'rich ground for exploring what tends to be taken for granted' - Echoes of Plato in recent mathematical education?

Despite the influences which Plato has had on the other philosophers, whom I have mentioned in the last section, it might still be questioned whether his theory has any value at all in justifying the study of mathematics today. It does not seem perplexing to the modern mind that a number can be a half and a double simultaneously. Even if the observation, that every sensible object is both a whole object yet has many parts, has allowed theorists to derive theories outside education, it might still be doubted whether observations of this kind provide the basis of a justification for the learning of mathematics. But if the modern adult mind does not seem to be perplexed that a number can be a half and a double simultaneously, a child might be. He or she might take paradoxes of this kind seriously and be in some way affected by them, so that mathematics might still provide some kind of vehicle for a special kind of thought. Only a few years ago a writer discussed a pupil's remark, in his rationale for learning mathematics, which casts some doubt on whether Plato's theory is completely bankrupt.

In a stimulating article, Davis (1995) firstly shows that the need for mathematics he is seeking '...should not be understood in the utilitarian terms of equipping children with the skills necessary for adult life, nor in the political terms of providing the understandings needed for democratic citizenship...' (p.6). His view is that mathematics can 'offer a rich ground for exploring what tends to be taken-for-granted' adding that 'we *need* to study mathematics to begin to understand our prejudices...'. Reference here to what is taken for granted and our 'prejudices' certainly seem to echo what Plato thought about the role of mathematics in what I have characterised as shaking oneself out of complacency. Moreover the similarity between Plato and Davis seems stronger when we consider the mathematical example which he uses to make his point.

Like Plato, Davis makes use of a simple point. It arises from a classroom episode in which children have been exploring the fraction 2/6 with practical equipment. Whilst some fractions, including 1/3, were deemed to be clearly equivalent to 2/6, when confronted with 3/9 one pupil, Jiema, charmingly remarked that 'it's not as equal as the others'. Davis reflects upon this child's remark and it certainly stirs *his* mind into action. He sees it as a confirmation that no matter how much one supposes that 'Western systems of logic are founded on a belief in the possibility of clean definitions, crisp edges, and unambiguous categories'. Things are much more fuzzy than we think. Unlike Plato, our prejudices are not ones which must be transcended in favour of the quest for an unchanging reality. Our prejudices are necessary because they are what make perceptions and actions possible. As he says: 'We *need* our prejudices - without them we would be unable to draw meaningful images out of the noise of sensorial possibilities that surround us' (p.5).

Mathematics comes into the picture, at this point, since it is responsible, it seems, for creating many of our prejudices. 'Rhythms and patterns that would otherwise go unnoticed are revealed through mathematics to be repeated in all forms of life at all levels of observation'. However, it seems that some modifications to mathematics are always possible and hence our prejudices are subject to revision. In this respect Davis gives the example that so many writers seize upon - the discovery of consistent alternative geometries. What conclusions he draws from the classroom episode are not altogether clear. It is as though in trying to suggest that fractions can have degrees of equivalence, the pupil is somehow revealing the essential contrast between fuzzy logic and mathematical logic. All he says about this episode with respect to the foregoing discussion on prejudices is:

> One way of interpreting this classroom event is to suggest that the participants were engaged in contrasting the crisp conclusions of one mathematical system with the fuzzy way we sometimes think. While not formulated in the established language of fuzzy logic, Jiema and the teacher had touched on a topic of current widespread discussion within the mathematics community. (p.6)

The fraction concept offers what Davis calls a 'site' for such an exploration. Such sites are '...possible locations for interrogating our mathematics – or, more accurately, for studying ourselves and the prejudices that shape our world.'

Despite the metaphysical differences, between Plato's theory and that which Davis has presented there are still significant similarities. The latter's theory begins with the same kind of elementary statements about mathematics which Plato believed could stimulate thought of a general sort. Moreover, both have used mathematics as a vehicle to challenge our conservative perceptions of everyday experience. Despite the fact that Davis would probably not look at matters in this way since he regards his approach to mathematics as 'enactive', the contemplative aspects of the subject still seem to be revealed to Jiema. The naivety with which one can say that '3/9 is not as equivalent to 2/6 as 1/3', seems to be very much of the same kind as saying that everything is one whole and yet not one whole, or that everything is a double and yet not a double.

The contemplative aspects of mathematics seem to be at work again in the theory which Davis sets out. The strangeness which sometimes arises in mathematics stimulates reflective thinking. Although he does not make it explicit, there is also the suggestion that what we might be achieving from this is a kind of self-knowledge. Recall that Sullivan (see page 92) believed, too, that 'mathematics is of profound significance ...because it exhibits principles that we impose. It shows us the laws of our own being and the necessary conditions of experience'. Davis does not make the further move of saying that mathematics is thus a fine art, but one of his points remains that certain insights into extra-mathematical issues are revealed through reflection upon mathematical activity.

Descartes and self mastery

Training the diverse mind by procedures *of unification*

If we return to Charles Taylor's illuminating account of the nature of reason, we find him arguing that Descartes had a radically different account of reason from Plato; one which, as we shall see, is characterised by *procedures*. He writes, '...in relation to Plato, Descartes offers a new understanding of reason, and hence of its hegemony over the passions, which both see as the essence of morality' (Taylor, 1989 p.144). Recall that for Plato being ruled by reason involved contemplating the order inherent in the Forms. But this view of reason was not possible for Descartes for whom, as Taylor reminds us, 'the universe was to be understood mechanistically', and this meant that what is out there to know is not directly attainable by a kind of acquaintance. In particular, as Taylor points out, this meant that:

> The account of scientific knowledge which ultimately emerges ... is a representational one. To know reality is to have a correct representation of things - a correct picture within of outer reality, as it came to be conceived. (Taylor, 1989 p.144)

This correct representation of things cannot however be *found*, it must be *built*, and Taylor adds that 'the order of representations has to be developed in such a way as to generate certainty, through a chain of clear distinct perceptions'. So that:

> The Cartesian option is to see rationality, or the power of thought, as a capacity to *construct* orders which meet the standards demanded by knowledge, or understanding, or certainty...If we follow this line... then the self-mastery of reason now must consist in this capacity being the controlling element in our lives, and not the senses; self-mastery consists in our lives being shaped by the orders that our reasoning capacity constructs according to the appropriate standards. (p.147)

On Plato's account mathematics allows us to turn towards the order of the universe. On Descartes' account self-mastery is supposed to be achieved by our 'capacity to *construct* orders' which becomes 'the controlling element in our lives' since it 'instrumentalises the desires'. To quote Taylor again: 'Reason rules the passions when it can hold them to their normal instrumental function. The hegemony of reason for Descartes is a matter of instrumental control' (p.150). For Descartes, the role of mathematics now is not as a source of paradoxes which stimulate the mind into reasoning and hence shake it out of complacency, nor is it a stepping stone which attempts to connect the objects of sense to those of knowledge. The role of mathematics is to act as the *model* for constructing the orders which 'meet the standards demanded by knowledge, or understanding, or certainty' (p.147). The shaping of our lives by such orders constitutes self-mastery. It is in this way that mathematics is supposed to provide mental training. To elaborate on this we need to take a closer look at Descartes' work. Just as we noted that Plato did not make much, if any, explicit use of the term 'mental training' so we must note that Descartes, too, does not speak of such training. However, we can see that at least one of his commentators does. It seems that Descartes wrote only twenty-one 'Rules for the Direction of the Mind' out of the thirty-six that he had intended. But of the first twelve Margaret Wilson (see Descartes, 1969 edn) explains that these were supposed to provide 'a general account of scientific procedure and how to *train the mind*...' [110]

---

[110] My italics

Descartes begins his *Rules for the Direction of the Mind* with an interesting distinction between the arts and sciences which we have already noted in the last section (see page 58). Recall that for Descartes sciences '...entirely consist in the cognitive exercise of the mind' whilst the arts '...depend upon an exercise and disposition of the body' (p. 36). Descartes distinguishes the sciences and arts in order to persuade us that they should not be approached in the same way. Whilst he agrees that a single individual could not practise *all* of the existing the arts, so that within our lives some kind of specialisation is inevitable, Descartes complains that we wrongly suppose that the sciences should be viewed in the same way, and studied separately. But for him all of the sciences taken together contribute to a particular goal, and so the first rule is that we should direct our attention not to the individual sciences but towards the special goal to which they are directed, namely 'good understanding' or 'universal wisdom'.

It is worth emphasising two points here. Firstly, in my discussion of Plato I compared what I called the 'initial conditions' that were supposed to hold before training with those of meditation, my simple example of mental training. In this respect, the initial conditions of Descartes' theory are different. Nevertheless, they do embody a 'tendency' of some kind, namely the tendency to treat the sciences as though they were like arts and to suppose that division of labour was essential in them. Secondly, the goal to which Descartes supposed all the different sciences of his day were directed, although different from the arts, was nevertheless a broad one, since for each individual it was concerned:

> ...not for the purpose of resolving this or that difficulty of scholastic type, but in order that his understanding may light his will to its proper *choice in all the contingencies of life*.[111] (p.37)

As it stands, then, Descartes appears to be providing us with an interesting rationale for a curriculum which promotes specialisation in certain particular 'arts', but which seeks some holistic approach to the 'sciences' based on a belief that some overall goal exists amongst them. The 'rules' that he set out, were supposed to guide, or more precisely 'direct', the mind to this overall goal and hence they provide a much more detailed mental training plan than that found in Plato's work. In what follows I shall draw out some of what I think are the salient rules.

---

[111] My italics

Firstly Descartes tells us that we must 'make it a rule to trust only what is completely known and incapable of being doubted' and since he thought that '...Arithmetic and Geometry alone are free from any taint of falsity or uncertainty...' (p.39), these must become the centre of attention. But Descartes makes it clear that we should not study mathematics to the exclusion of other 'sciences', but that

> ...in our search for the direct road towards truth we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of Arithmetic and Geometry'. (p.40)

There are two 'mental operations' in mathematics which Descartes thinks are responsible for guaranteeing certainty, namely, 'intuition' and 'deduction'.[112] He gives an account of what he thinks is a special meaning for the first of these two operations. It is, he says, 'the undoubting conception of an unclouded and attentive mind, and springs from the light of reason alone' One example he gives is that 'a triangle is bound by three lines only'. By *deduction* on the other hand he means '...all necessary inferences from other facts that are known with certainty' (p.43).

It is tempting to think that intuition concerns only the truth of the premises of a deductive argument and that the step by step procedure is the more important one. But Descartes points out clearly not only that:

> ...none of the mistakes that men can make...are due to faulty inference; they are caused merely by the fact that we found upon a basis of poorly comprehended experiences, or that propositions are posited which are hasty or groundless. (p.39)

So that *intuition* is required throughout '... discursive reasoning of whatever sort'. The example he gives is helpful in reminding us of the importance of not just *using*, what we now call 'transitivity', in deductions, but that we must *see*, by intuition, that this relationship must hold in certain cases, especially that of equality. He remarks that if we want to show that '2 and 2 amount to the same as 3 and 1', it is not sufficient simply to see that '2 and 2 make 4' and '3 and 1 make 4'. We must also see, intuitively, that the required conclusion follows from them.

---

[112] I had reason to introduce the notion of 'intuition', in one of its senses during my discussion on the central role of perception in art. There, intuition was used in a sense which made it susceptible to being overturned by mathematical thinking. For Descartes intuition means the grounds of certainty.

The importance of intuition and deduction as the means of establishing truth needs further elaboration. It is certainly true that some 'facts' seem to be intuitively true. It is also true that inferences can be made from known facts to derive new facts. However, if this is an account of the correct order of deriving facts then in general it is false. In mathematics it is indeed possible and common to begin with a fact, often in the form of a definition and to infer an indefinite number of other facts. However, it is also the case that many facts which can be *validated* by such deductions from intuition are not necessarily originated in this way. If this were the case then proving established results in mathematics would not be as difficult as it often is. Often the result is already believed to be true and the difficulty is in convincing oneself and others that it really is true. This may involve working backwards from the result to the known facts, or working forwards from the fact and backwards from the result at the same time, gradually closing the gap between these chains. More often the deductive steps required to show that something is true are not themselves established deductively. Imagination, experience and insight all play their part. So, Descartes' remarks seem to be more concerned with the logical status of our reasoning than with the psychological.

Descartes tries to get closer to the core of mathematics, to what he calls 'universal mathematics', by reducing it to just two elements with which it is concerned, namely, 'order' and 'measurement'. He then proceeds to discuss the method for arriving at his goals which are, in accordance with what I remarked earlier in my account of training (see page 129), namely, trying to get thinking to go along certain lines different from usual tendencies. In tackling any enquiry it seems we should bring to bear a certain order on its elements. This involves taking simple matters first and familiarising ourselves with these before using them to move on to the more profound. Descartes gives us ways of ordering the task in respect of what he calls 'relative' and 'absolute' aspects and it is necessary that we can follow through any of our claims in a continuous 'movement of thought'. So sufficient reference is made to *order* to prevent Descartes supposing that the truth unfolds to the thinker in the same way in which it is later written up.

Descartes makes frequent reference to mathematics but the *learning* of mathematics is not explicitly connected with mental training until we reach Rule X where he does make an explicit educational point. The foregoing rules show how impressed he is by the way knowledge is validated but not precisely how it is arrived at. He summarises Rule X in the following way:

> In order that it may acquire sagacity the mind should be exercised in pursuing just those inquiries of which the solution has already been found by others; and it ought to traverse in a systematic way even the most trifling of men's inventions, though those ought to be preferred in which order is explained or implied. (Descartes, 1969 edn p.67)

Descartes proudly claims that whenever he read a book which 'promised some new discovery' he would try and anticipate this discovery before reading the published account. Unlike many others it seems he was successful and he concludes that this was a particularly worthwhile approach and one for which he developed certain strategies. Whilst he does not outline these personal strategies of his, he does suggest that others develop in a similar way. He does realise that not everyone will succeed since the books he tackled would not be appropriate for everyone to tackle. However, he suggests a range of activities which one should engage in. Once again the notion of order appears in his choice of exemplary activities. The activities should not necessarily be difficult ones, but those, which ought to be undertaken, should display order. He suggests weaving and embroidery alongside 'all play with numbers and everything that belongs to Arithmetic, and the like'. About such activities he says:

> It is wonderful how all these studies discipline our mental powers, provided that we do not know the solutions from others, but invent them ourselves. For since nothing in these arts remains hidden, and they are wholly adjusted to the capacity of human cognition, they reveal to us with the greatest distinctness innumerable orderly systems, all different from each other, but nonetheless conforming to rule, in the proper observance of which systems of order consists the whole of human sagacity.[113] (p.68)

It is not quite clear what kind of arithmetical activity would be appropriate here. In his reference to rule-following Descartes could be recommending the following of a routine *algorithm* such as, for example, long division. This would reveal an orderly system of a kind, and there are those today who believe that such practice is a form of mental discipline. This would be in keeping with the other activities in his list. Weaving and

---

[113] Even though Descartes, earlier, speaks about *traversing* in a '*systematic* way', thus drawing out the procedural aspects of mathematics and other activities, there are still some traces of a contemplative attitude to these rule-following systems since they are *revealed* to us in certain activities which *constitute* human wisdom.

embroidery are routines only a little different, in this respect, to arithmetical routines.[114] Moreover, much of arithmetic appears to have been taught by rote in and around Descartes' time, so the idea of working out a solution by oneself rather than being able to reproduce a teacher's example would be some innovation. On the other hand, what Descartes has to say, in the same work, about how the use of syllogisms can result in lazy thinkers suggests that the position on this is not clear. We shall see later on, that even though the belief that mathematics can train the mind is still upheld in very recent times, the *kind* of mental activity necessary for such training is still somewhat in doubt, even though the belief that mental training is a fit purpose for mathematics still exists in recent documents. Nevertheless, if this is all Descartes has in mind, then his notion of mental training does seem to conflict with the point at which we started, namely, to explore contemporary remarks from such sources as the HMI booklet (see above page 127). There, recall, mental training seemed to involve the 'higher functions of the mind' rather than 'stereotyped working to rule'.

In summary, Descartes offers us two aspects of mental training. The first is based on the way mathematics is validated. All our wisdom, it seems, ought to be validated in this way. However, what he calls the 'mental operations' of 'intuition' and 'deduction' cannot simply be adopted as ways of thinking, without considering that the order in which the string of reasoning is carried out in mathematics, depends upon imagination and experience. Descartes does discuss ordering and analysis, but it is not really of the kind that relates directly to mathematics, even if it does support the view of training as trying to correct a general tendency amongst people to think in less fruitful ways. The second aspect is much more explicitly linked to carrying out mathematics, though it seems that other activities will also suffice, but nothing he says prevents us from concluding that what he is suggesting is the practice of routines.

Plato and Descartes compared

For both Plato and Descartes the nature of mathematics permits mental training. For Plato, to achieve wisdom it is necessary to reflect upon the nature of mathematics - its

---

114 A slightly different reason for providing opportunities for following simple procedures, beginning at an early age, has been set out by Anita Straker (1986). She argues that the dominant role that programmable machines are having in our lives demands a general facility with procedures.

apparent paradoxes of quantity, for example, and its paradoxical relationship with the sensible world - in order to enable the mind to apprehend the Forms, and in particular the Form of The Good which constitutes the Order upon which our moral life is based. What is for Plato fundamentally a contemplative business is for Descartes more discursive. To achieve wisdom, the later thinker speaks in terms of a procedural kind of order. One constructs order by following a process similar in kind to the way mathematics is validated, or more mundanely we apprehend the notion of order by undertaking exercise or tasks of a variety of kinds which are rule-following. These include arithmetic but other activities are included too. Plato urges us, or at least urges those who will educate his philosophers, to seek activities in which our 'minds are drawn from the world of change to reality' as opposed to 'a training in habituation'. Descartes seems in some respects to favour habituation.

We have seen that the *procedures* involved in mathematics are for Descartes at the heart of the mental training process, both in the more general sense of constructing order, and in the specific cases of undertaking activities which reveal 'innumerable orderly systems'. The idea that reason is somehow embodied in mathematics persists in more recent times. But we need to question whether this notion of reason has been subjected to further changes in more recent accounts. Taylor, it will be recalled, distinguished between the substantive and procedural accounts of reason in Plato and Descartes, and made a distinction between reason *found* and reason *built*. In more recent accounts, reason is sometimes directly linked to *logic*. So that mathematics is either explicitly linked to training the mind in logic; or training in logic and training in reason are treated as though they were synonymous.

The influence of Descartes' procedural view

Fitch – Mathematics as training in *logic*

The value of what I am calling procedural reason, after Charles Taylor, was clear in the work of Sir Joshua Fitch. In his *Lectures on Teaching* given at Cambridge during the latter part of the nineteenth century he argues without any caution that '...the fundamental reason for teaching mathematics at all either to boys or men' is because

> ...a certain kind of mental exercise, of unquestioned service in connexion with all conceivable subjects of thought, is best to be had in the domain of mathematics. (Fitch, 1902 p.342)

The boldness of his view, which is worked out in some detail, makes Fitch an exemplary proponent, in the last hundred years or so, of the view that a rationale for learning mathematics may be based on its fitness for providing mental training. I have mentioned his work earlier when it was noted that he believed that mathematics, or arithmetic at least, could be viewed as either an art or a science. The former view, as we saw, focused on the utility of the routines of the subject, and Fitch was, perhaps unusually for his time, hardly impressed with this view. But it is within his view of mathematics as a *science* that he believes mental training is possible.

For Fitch, arithmetic is a science when it 'investigates principles' and because,

> ...he who unearths the truths which underlie the rules of Arithmetic, is being exercised, not merely in the attainment of a particular kind of truth about numbers, but in the processes by which truth of many other kinds is to be investigated and attained. (p.287)

His belief in the value of the procedures of mathematics is clearly signalled by his use of 'processes' of finding truth. The importance of this aspect of arithmetic follows from Fitch's premise that the 'main purpose of our intellectual life is the acquirement of truth', by which he does not mean the collection of isolated facts. Indeed Fitch is concerned with what we might call *his* scientific approach to knowledge and truth, which requires:

> the recognition of every separate phenomenon in the shifting panorama of life as an illustration of some principle or law, broader, higher, and more enduring than itself....every particular fact worth knowing is connected with some general truth, and it is in the tracing of the connexion and collocation of particular and separate truths with general and abiding truths that science mainly consists. (pp.315-6)

This is perhaps a reasonable position to take on the facts of science, but Fitch extends this idea to *historical facts* and *grammatical rules.* Whilst we might be prepared to agree that any particular rule of grammar has 'scant meaning or use for us until it is seen as part of the science of language, it is highly questionable whether historical facts are similarly bound to a kind of science of history. Yet Fitch's scientific approach clearly does embrace other areas of knowledge, so that each historical fact is, '...to little purpose unless it is seen in its bearing on some political, economic, or moral law'. With this overarching view, the correct educational approach is clear. Pupils must '...learn to see special facts and bits of experience in the light of the larger generalisations by which the world is governed and held together'. (p.316)

His use of 'in the light of the larger generalisations' is deliberately indeterminate. It might mean that pupils must see how isolated facts somehow establish, inductively, certain general principles, or it might mean that pupils must learn to see how isolated facts are derived from the principles. Fitch is aware of the value of both induction and deduction. Indeed at first his plea is that pupils should be competent in *both* kinds of reasoning:

> ... it is a great part of the business of education so to train the faculties that whichever process we adopt we should use it rightly, that our generalisations shall be valid and sound generalisations, and that our inferences shall be true, not hasty and illegitimate inferences, from the facts which may come before us. (p.317)

Moreover he points out that although inductive reasoning is most typical of the physical sciences, arithmetic and geometry use both kinds of reasoning. But to secure his strong claim for mathematics, he has to restate his commitment to the method of deduction, which he insists is 'the characteristic mode of procedure in arithmetical as well as in all other departments of mathematical science'. Fitch adds some other remarks about the nature of mathematical 'truths', which he says:

> have the great advantage of being very simple and very evident. They lie quite outside the region of contingency or controversy, and they therefore furnish a better basis for purely deductive or synthetic logic than any other class of subjects in which the very data from which we proceed are often disputed, or at least disputable. (p.318)

So, in true Cartesian spirit mathematics is a science because it is deductive, and has truths that are at once simple and evident and certain. But, recall, Descartes sought a procedure which would be of use for 'all the contingencies of life'. All that remains for Fitch to do, is to make good his claim that mathematics embodies the dominant model of reasoning for 'all conceivable subjects of thought', for his view to be barely distinguishable from what Descartes had believed earlier. But to approach this position, he must restate the important point, now somewhat eroded by his admission of the different kinds of reasoning, that mathematical reasoning is of *general* application. This he does in the following way:

> Suppose then I want to give myself a little training in the art of reasoning; suppose I wish to get out of the region of conjecture or probability, free myself from the difficult task of weighing evidence, and putting instances together to arrive at general propositions, and simply desire to know how to deal with my general propositions when I get them, and how to deduce right inferences from them; it is clear that I shall obtain this sort of discipline best, in those departments of thought in which the first principles are unquestionably true. (p.319)

From here he feels that he is justified in asserting that '*...the proper office of arithmetic is to serve as elementary training in logic*'.[115] (p.321)

Although I have argued that Fitch has more in common with Descartes than Plato, that is not the way that Fitch would necessarily have seen himself. He aligns himself with the earlier man when he remarks how Plato and his followers valued rigorous mental training through Geometry in their quest for the solutions to a range of moral questions. As he put it, these thinkers thought

> ...that a man whose mind had not undergone a rigorous training in systematic thinking, and in the art of drawing legitimate inferences from premises, was unfitted to enter on the discussion of these high topics; and the sort of *logical discipline* which he needed was most likely to be obtained from geometry – the only mathematical science which in Plato's time had been formulated and reduced to a system.[116]
>
> (p.320)

If this were true then there would seem to be little of difference between Plato and Descartes in respect of the educational value of mathematics, or at least there would be little between them if logical rigour were the only reason Plato valued mathematics. What I have been trying to draw out, is that as far as *The Republic* is concerned, which contains some of the most detailed accounts of Plato's view of education, Plato has emphasised the more reflective value of mathematics in education; the logical rigour of mathematics is not of primary educational importance to him. What is of more importance, are the paradoxes and puzzles of mathematics, and the ontological status of the premises upon which mathematical reasoning is based.

---

[115] My italics
[116] My italics

The value of logical training through the learning of mathematics – a criticism

There are two main criticisms of Fitch's view. Firstly, recall that Fitch's view is premised on his main purpose of intellectual life as the acquisition of truth, and that all valuable facts are connected with general truths. So that the main value of learning mathematics lies in the assistance it can give us in drawing out facts from general principles, or what Fitch calls 'deducing right inferences'. If examples could be found of several kinds of worthwhile facts, which are not necessarily connected with general true principles, or at least not *deductively* connected with any such principles, then this would severely weaken the case that Fitch makes. With such examples we would have shown that all Fitch could have rightly claimed, would have been the value of mathematics in certain kinds of scientific knowledge, since some of this knowledge is surely based on inferences made from law-like principles of a general nature. This would greatly reduce the scope of Fitch's claim that mathematics trains the mind. Any thoroughgoing rationale for the subject would then have to point out that mathematics provides a training which will be of service to certain areas of life, and not to 'all conceivable subjects of thought' (p.342) as Fitch believed.[117]

It is easy to sympathise with Fitch, in his insistence that facts must be connected to principles, especially if we consider that many of the teachers of his day might have resembled Charles Dickens' Gradgrind in 'Hard Times' for whom facts were given an exaggerated status. It is to the credit of educators like Fitch that issues surrounding the learning of mere facts are just as alive today as they were in his own time. Nevertheless, a thoroughgoing case, such as Fitch tries to make, is not in the end sustainable. John Passmore, in defending the value of imparting information in teaching, finds himself confronted with views, some of which are not unlike Fitch's, which stress the primacy of principles over facts. But Passmore's reply to this is very instructive. In many contexts, he remarks, principles are non-existent:

> ...there is no principle from which it can be deduced that Henry VIII broke with the Church of Rome, or how rapidly the population of India is increasing, or how many ribs a human being has, or when President Kennedy was assassinated, or how distant the moon is from the earth, or, even, that chloroform is an anaesthetic or that platypuses suckle their young. (Passmore, 1980 p.97)

---

[117] Aristotle believed in the value of principles but he also pointed out the value of experience see The Metaphysics

But it would be absurd to deny the value of this information simply by showing that it is not the result of some inference from principles. Indeed matters are worse than this. Passmore points out that in some contexts there are real dangers in assuming the existence of principles:

> ...the mistakes in American foreign policy since the Second World War largely derive from a too powerful tendency to think conceptually and a weak appreciation of historical factors. Similar considerations apply, scarcely less obviously, to the practice of medicine or engineering; to think entirely in terms of general principles, without respect for the peculiarities of individual cases, is the path to disaster. (p.102)

Hamlyn (1967) says essentially the same thing:

> Explanatory theories, e.g. scientific theories, have a logical structure in the sense that the propositions of which the theory is constituted can be arranged in a certain order, so that certain propositions can be derived from others. It is indeed the fact that from general laws and statements of initial conditions it is possible to derive conclusions – the facts to be explained – that provide the basis of scientific, and no doubt other kinds, of explanation. (p.28)

But Hamlyn is more careful than Fitch to point out that:

> This...only applies where the discipline in question constitutes a theory – to parts of science, the foundation of mathematics and so on. Where questions of explanation do not arise and do not have a place, then this sort of consideration has no place either. It is difficult to see how large parts of history or literature could be said to have a structure in this sense. (p.29)

Indeed Hamlyn argues that in learning there must always be a 'delicate balance between principles and cases'. Both Hamlyn and Passmore have distinguished between different kinds of knowledge. Where Hamlyn is careful to distinguish subjects which are theoretical from those which are not, Passmore prefers to distinguish between open / closed, broad / narrow capacities. For him scientific knowledge is open and broad, and in this way is distinct from historical knowledge which is narrow yet still valuable.

We should note with Fitch, just as we did with Descartes, that any valid proof for a theorem must follow deductively, but that this does not mean that anyone who was able to carry out such deductions would have been able to have formulated the proof. It is one thing to *follow* a proof, quite another to *originate* one. Originating proof involves deductive thought but also a good deal of imagination, insight, experience, trial and error, and awareness and expertise in useful strategies. A machine can in principle carry out deductions but not all proofs.

So whilst Fitch is right that deductive method is typical of mathematics, he does seem to underplay other ways in which mathematics proceeds, even though he does suggest that inductive reasoning plays some part. However, having conceded this much, Fitch does rather overstate the notion of *truth* in mathematics. There certainly was a time when it was thought not only that Euclid's geometry was consistent, but that also that it described the space in which we live. The discovery of alternative geometric systems have shown not only that this cannot be true, but that other systems describe space more appropriately. Such observations as this have led to the view that mathematics does not describe the world, rather it consists of models which are constructed securely and consistently, but their truth arises from inner *coherence* rather that *correspondence* with some extra-mathematical reality. There is a price to pay for attempting to match up a theory with the world - an axiomatic system free from doubt, but free from substantial content.

Charles Taylor helped us to see that Plato's and Descartes' views were linked with the outcome of self-mastery. Fitch gives us no indication that he has moral ends of this kind. The most that logic can provide, it seems, is a certain kind of freedom. But this is not freedom from the passions, but simply freedom from 'the difficult task of weighing evidence, and putting instances together to arrive at general propositions'. So the notion of reason has changed its meaning from Plato, through Descartes, to modern times. Nevertheless, it is still supposed by many, Fitch in particular, that the logical reasoning of the kind found in mathematics remains of universal value.

Toulmin's critique of 'idealised logic'

A more sustained and penetrating criticism of valuing what he calls 'idealised logic' can be found in the work of Stephen Toulmin. One of Toulmin's aims in his book *The Uses of Argument* is to reveal two models of reasoning: one based on the procedures of jurisprudence [118] and the other based on formal logic that has mathematics as its model. He writes:

> From the time of Aristotle logicians have found the mathematical model enticing, and a logic which modelled itself on jurisprudence rather than geometry could not hope to maintain all the mathematical elegance of their ideal. Unfortunately an idealised logic, such as the mathematical model leads us to, cannot keep in serious contact with its practical application. (Toulmin, 1958 p.147)

One of the main points of Toulmin's argument is the nowadays fairly established view that discourse takes place in a variety of non-reducible contexts. Often influenced by Wittgenstein, a range of theorists have insisted that there are different kinds of discourse and hence knowledge and facts. In education, for example, Hirst (1965) has famously attempted to delimit kinds of knowledge on the basis of their distinctive concepts and methods of validation. More recently, some of the theorists of the 'critical thinking' movement have also been quick to deny the existence of general powers of thought. But I have chosen to outline some of the main points of Toulmin's argument, particularly since it directs its attack at the primacy of the kind of logic which has been explicitly linked to mathematics by Fitch.

Toulmin's argument, as the title of his book suggests, involves a consideration of the nature of argument itself. Arguments, for Toulmin, are at root, ways of supporting claims and assertions when such support is required, as it always is in principle. Every day of our lives we make claims and assertions. But these are not arguments. Arguments give what Toulmin calls 'backing' to such claims and assertions when required. The wide variety of contexts in which arguments take place are what he calls 'fields of argument'. In assessing arguments Toulmin asks 'what things about the form and merits of our arguments are *field-invariant and* what things about them are *field dependent*?'

One kind of claim, which arises in a variety of different contexts, is the claim that something *can or cannot* be done. For example one cannot lift a ton, one cannot call a king 'she', one cannot ride a bike on the pavement etc. The arguments that are provided

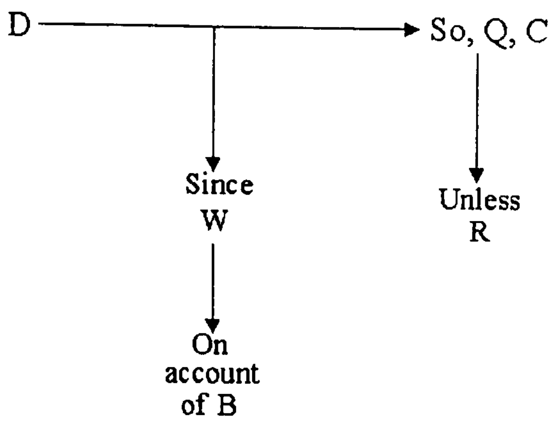
---

[118] See the last chapter of Kline (1972) for an account which, against the spirit of Toulmin's position, *does* attempt to show that law may be regarded as following similar lines to the axiomatic approach of mathematics.

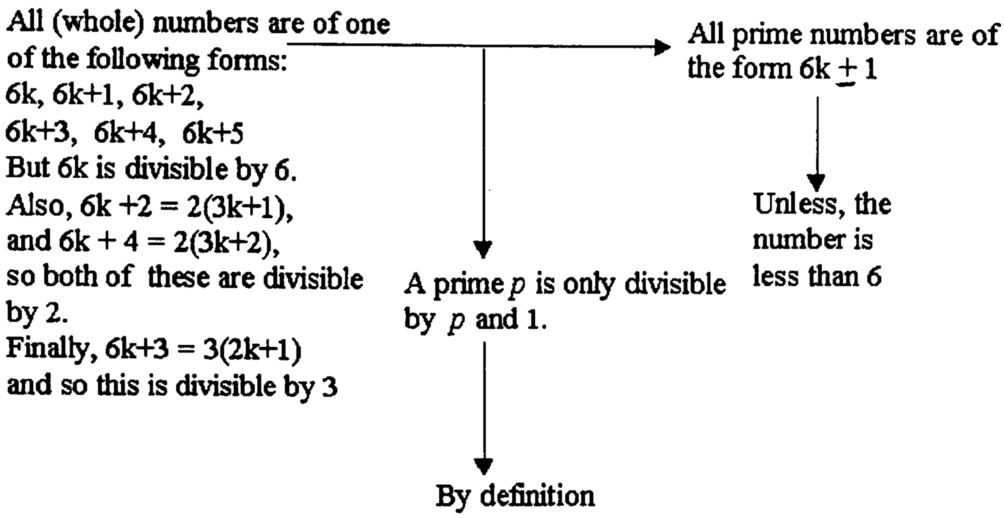
in support of these claims vary with respect to what Toulmin calls the *criteria* for the term 'can', and in this respect such arguments will be field dependent. What can be done in algebra will be different from what can be done within the law. But what he calls the *force* of 'cannot' as an injunction, a means of ruling out something, is *field-invariant,* the disregarding of which involves a consequence of some kind - incoherence, penalty, contradiction. *What* is ruled out will vary from field to field but the fact that something is always ruled out is field invariant. The fact that some consequence follows from ignoring this injunction is also field invariant, though the kind of consequence will be field dependent. Later on he generalises this point in saying that 'all the *canons* for the criticism and assessment of arguments… are in practice field-dependent, while all our terms of assessment are field-invariant in their *force*'.

So far then, if mathematics is to provide any general training in reasoning, in arguments at least, it must be in terms of training in *force* rather than *criteria.* One must learn, that is, that terms like 'can' or 'possible' have the same force in contexts outside mathematics as they do in mathematics itself. But the field invariance of 'modal' terms can equally be gained from other fields and therefore mathematics has no special claim to develop the mind.

Toulmin investigates next, the layout of arguments. When an argument is a means of justifying an assertion or claim, such claims form the conclusion of an argument. But if the conclusion is to be more than simply an assertion, it will rely upon some backing in the form of *data.* Not any data will do, of course, it must be data which is appropriate to the conclusion in question. Furthermore a *warrant* is required, which allows the conclusion to follow from the data. In some cases there will be exceptions to a *warrant* and the *qualifier* indicates what Toulmin calls 'the strength conferred by the warrant on this step'. Finally, the warrant, itself may be called into question. It is always a warrant on account its own backing, and there are in principle always cases where the warrant may be overridden and this is marked out by a rebuttal R. So that we have the following layout with Data D, Conclusion C, Qualifier Q, Condition of rebuttal R, warrant W and the backing of a warrant B:



What Toulmin claims is field dependent here is the backing needed to establish our warrants. We can give a mathematics example here:



So in learning mathematics one might use a common layout especially where proof is required.

The standard syllogism has much in common with this layout.

A is a B
All Bs are C
A is a C

The first line is a data line, the second apparently a warrant and the last line a conclusion. But Toulmin believes such paradigms of logical form mask a more complex situation in everyday reasoning. He suggests that the second line can be taken as either a warrant or the backing for a warrant, especially if second lines are of the form, 'Scarcely any As are B', or 'Nearly all As are B', in which case the second line is backing and therefore it is field dependent.

One of the most important distinctions for Toulmin is that between a warrant and its backing. He wants to show how the idea of logical form is unhelpful. Any statement can lead to any conclusion provided a suitable warrant is inserted. But not all warrants are accepted and the backing that is needed is field dependent. In the above mathematical example the argument is valid because the warrant guarantees it. If a number has a divisor of 2, 3, or 6 it cannot be prime. So primes must be of the only remaining forms. But the warrant here is backed by definition, that is to say agreement, within the mathematical community. So it seems that mathematics provides no special training beyond the use of form in this minimal sense, and if Toulmin is right every other subject can train the mind in this sense. To be able to argue we must get down, at some point, to the level of types of backing, and mathematics can only provide a limited source of these, particularly definitions.

Toulmin thus provides us with a powerful objection to any rationale for learning mathematics, along the lines which Fitch's runs, which claims that the kind of reasoning acquired through the study of mathematics is of widespread or even universal application. However, not all mental training theorists who have stressed the importance of the procedures of mathematics have believed that these procedures are of universal importance. Indeed some have emphasised the dangers of supposing that such procedures can be transferred to other contexts. These theorists, then, seem to provide a more modest view, and we shall now consider what they have to say on the matter.

Chapter 10 – A more modest account of mental training

Initiating the reasoning powers through mathematics – Tate and his legacy.

It is been interesting to note how mathematics is seen to be valued, by some writers, on account of its ease of accessibility, whilst for others the sheer inaccessibility of the subject is what is at issue. For example, we saw in my discussion of the connection between aesthetics, art and mathematics how the aesthetic properties of mathematics were regarded by some (see Leibeck, 1984) as the main reason why even young children pursued mathematics. Others like Poincaré were in some way sceptical about the universality of aesthetic capacities, at least as far as mathematical discovery is concerned. This in turn was denied by Papert who was convinced that aesthetic processes could regulate the mathematical thinking of those who were given certain opportunities. Huntley's remarks seem to lie somewhere between these extremes, though the particular examples which he gives are all set in quite advanced mathematics.

A similar ambivalence surrounds the basis that mathematics can train the mind. Locke, as we saw in Part 1, believed that Arithmetic 'is the easiest and consequently the first sort of abstract Reasoning, which the Mind commonly bears, or accustoms itself to'. Fitch, too, as we have just seen, thought that since mathematics is simple and 'very evident' it could 'furnish a better basis for purely deductive or synthetic logic than any other class of subjects'. Isaac Watt, on the other hand, who was not an unequivocal defender of mathematics, believed that the sheer *difficulty* of mathematics provided a good reason for at least being exposed to it. In a similar manner to Descartes, he sets out rules for improving the mind, but in one respect at least his approach is radically different. One rule involves teaching oneself *humility*. Watt (1809) writes:

> You should…contrive and practise some proper methods to acquaint yourself with your own ignorance, and to impress your mind with a deep and painful sense of the low and imperfect degrees of your present knowledge, that you may be incited with labour and activity to pursue after greater measures.

Whilst, he does not suggest that all mathematics is suitable for this purpose, since compared with the sciences much of mathematics leaves 'scarce any doubt', he urges his reader to:

> Spend a few thoughts sometimes on the ...doctrine of infinites, indivisibles, and incommensurables in geometry, wherein there appear some insolvable difficulties: do this on purpose to give you a more sensible impression of the poverty of your understanding, and the imperfection of your knowledge. This will teach you what a vain thing it is to fancy that you know all things, and will instruct you to think modestly of your present attainments, when every dust of the earth, and every inch of empty space, surmounts your understanding, and triumphs over your presumption.

This exposure to tortuously difficult aspects of mathematics, however, needs to be mentioned only so that it may be set aside for the moment. It is to those who are convinced of the simplicity of essential aspects of mathematics that I now want to turn.

There is, as I have begun to show, a stream of thought that has elevated mathematics over other subjects because it provides a certain kind of training in reason by dint of its simplicity, or what I shall call later its 'transparency'. A third, modest, theory of mental training emerges out of a consideration of the supposed simplicity of mathematics. Procedures are still important in this theory. But what distinguishes this modest form of training from those such as Descartes and Fitch is that the particular mathematical procedures that are learnt are not supposed to be generalised across other subject areas as a kind of master key. What is important is simply the disposition of the trainee to raises questions, give reasons and use appropriate evidence. Mathematics, therefore, is supposed to provide the simplest *initiation* into rational procedures. The same view, as we shall see, is sometimes carried over into more explicit social settings and in extreme cases to a moral or at least a quasi-moral context. What is transferable to other areas of life is not so much a particular generalised *skill*, but more an awareness or understanding of what characterises rational discourse. It is seeing that there is a kind of rational etiquette. Although this view can be traced back to St. Augustine, again, I shall begin with the nineteenth century educationist Thomas Tate, who has discussed the view in some detail.

Half a century before Fitch, who in 1880 gave his 'Lectures on Teaching', claiming as we have seen that the main value of mathematics was as a form of logical training, an influential work *The Philosophy of Education* was published by Thomas Tate. In this book, Tate devotes a section to the 'cultivation of reason and judgement', both of which he begins by running together, describing them as 'a mental faculty whereby we distinguish truth from falsehood'. From then onwards he focuses on the notion of *reason*. The educational ideal of *self-mastery* which is made explicit in Plato, but which is not mentioned in Fitch, emerges again in Tate. He writes:

> Reason in a well regulated mind, holds the *mastery* of all the other faculties: it gives strength and precision to every one of them, and harmonizes and regulates their operations as a whole...[119]
>
> (Tate 1854, p.88)

There are, here, echoes of Plato's view of the mind as involving parts or faculties, which have a tendency to conflict and are in need of harmonisation. Yet for Tate, whilst some mention is made of the *passions*, he prefers to contrast reason with *imagination*. Clearly, this cannot be the same imagination which we saw in Part 2 was invoked in support of the claim that mathematics was a fine art. It is rather more like what some writers have called mere *fancy*.[120] Thus Tate writes:

> When we neglect the cultivation of the reason of young persons, their minds become engrossed by trifles, or carried away by the wild freaks of imagination; and the most sacred and momentous opinions are either treated with unbecoming levity and indifference, or accepted without thought or reflection. Such persons readily become the victims of sophistry, or the willing slaves of superstition and bigotry. (p.88)

The similarity between Plato and Tate aside, there is at least one notable difference between them. Whereas for Plato the apprehension of reason was characterised as a long and difficult task, and was thus reserved for the few, Tate insists that the development of reason is more universally attainable. He writes:

> No faculty in our nature is more susceptible of cultivation than reason; and the neglect of its cultivation is attended with the greatest possible evils, as well to the individual, as to society at large. No doubt, there are original differences in the power of reason or judgement, but we have no hesitation in stating, that the chief source of the differences in this power, found amongst men, is to be traced to culture and regulation. (p.88)

Given, then, that reason does provide some self-mastery - that it prevents the individual from being at the mercy of his or her imagination and the domination of others – given too that in varying degrees it is universally attainable, what contribution can

---

[119] My italics. It is perhaps worth pointing out, that although Tate is emphatic about the impossibility of using mathematical procedures across other *subject areas*, he still sees reason as unifying within the individual.

[120] See Passmore (1980) Ch.8.

mathematics make to its acquisition? Tate asserts that: 'the reasoning powers of a child are exercised whenever we put the question *why*, or because'. Whereas for Plato the 'why' and 'because' ultimately depended upon an acquaintance with the Forms, Tate's demand here is more modest. He seems less concerned than Plato and Descartes with finding or constructing the *source* of reason, but more with its *practice*. For Tate the reasoning faculty develops, in the first instance, as pupils begin to understand the *necessity* of giving grounds for truths, where appropriate, and responding to 'why' questions with appropriate 'because' answers. Like Fitch he does acknowledge the existence of *principles*, but these do not have a central role in his theory. As he remarks '...the higher principles of a science should never be taught before the pupil has been made acquainted with the relations and analogies of the most familiar facts' (p.90). Thus it is with more basic elements of training in reason that he is concerned. But it is important to note that even at a later stage these 'higher principles' do not become the source of *general* logical principles that they were for Fitch.

For Tate, just as it was for Descartes and Fitch, it is the *simplicity* and *certainty of* mathematics which make it such an ideal vehicle for developing the reasoning powers. Thus, in outlining these essential properties of mathematics, Tate writes:

> 1. Nothing is taken for granted or on mere authority; for its principles of reasoning are axioms or self-evident truths.
>
> 2. Its proper objects are the relations of numbers, lines and spaces, things which are cognisable by our senses, and which can be defined and measured, with a precision of which the objects of no other kinds of reasoning are susceptible (p.80).

Since Tate's conception of mathematics is so close to that of Descartes' and Fitch's, it is worth emphasising the distinction between Tate's modest role for mathematics in the development of reason, and the stronger claim that mathematical training is of *universal* application. Recall that Descartes supposed that the principles of mathematics were the best model of reasoning for '*all* the contingencies of life'.[121] Similarly, Fitch argued that mathematical reasoning unquestionably serves '*all* conceivable subjects of thought'.[122] Tate, however, is sceptical about the direct transfer of mathematical reasoning to all other areas. This is a very important point to stress and it is upon this very point that Tate's view is a modest one. He remarks:

> Although the mathematical sciences may form one of the best initiatory trainings of the reasoning powers...it only exercises the mind in appreciating one kind of evidence, - namely, *mathematical evidence.* Some other subject, therefore, should be adopted for the purpose of developing the reasoning powers of children in relation to *moral evidence.* (p.90)

In saying this, he is anticipating views in recent years, such as Toulmin's critique of the value of 'idealised logic' which I have already outlined, and those such as Hirst's theory which argue that knowledge is of different kinds, each one embodying its own methods of validation. Such views are often supposed to be conclusive in showing that some kind of general reasoning power cannot be achieved outside a particular enterprise. So that Descartes' project of employing the 'universal mathematics', and Fitch's general logical training for use across the areas of thought, are in the end futile.

But Tate does not seem to be open to this kind of criticism. His important point seems to be that mathematics is valuable in *exemplifying* certain aspects of a rational enterprise, that of giving reasons, or more precisely of providing evidence to support a claim. But his insistence upon the limitation of the kind of evidence involved – mathematical evidence – anticipates, as I have already suggested, Toulmin's critique. In particular Tate would agree with Toulmin that arguments are 'field dependent' with respect to the evidence that is appropriate in such aspects of an argument as Toulmin refers to as 'backing' and 'warrants'. But Toulmin did admit that some aspects of arguments are 'field invariant', and for Tate to preserve some kind of theory of training he needs to show how mathematics has *some* field invariants. What are they?

The answer to this question seems trivial. It is simply that what is common to all arguments is the necessity for warrants and backing *of some kind.* But Tate's point seems to be that the backing used in arguments outside mathematics is not only often less accessible, but also that it relies upon *authority.* It is thus not as simple, self-evident, and thus certain, as the backing in mathematics. It is in this way that it provides an example of *a* rational enterprise. It has what I am calling its own 'rational etiquette'. But in being a particularly helpful exemplification of how reason-giving is essential in a range of areas of life, it has nothing to show about the particular kinds of reasons that are appropriate in other enterprises, only that reasons, of some kind, will be appropriate

---

[121] My italics
[122] My italics

there. In learning the subject it seems we can come to learn the appropriateness of reason-giving, and thus become disposed to seek and supply reasons. Much, of course, will depend upon the kind of teaching involved, but in principle Tate's position seems both distinct from the procedural accounts examined already and also somewhat plausible.

He elaborates further on the limitations of the kind of reasoning embodied in the 'mathematical sciences', and also the danger of supposing that such reasoning can be of general use across a range of human discourse:

> These branches of knowledge may give a false direction to the mind, if they are not taught with caution, and in connection with *moral science*. The certainty, and peculiar nature, of mathematical science, often inspire the disposition to demand the same kind of demonstration in other points. (p.90)

In order to make his point even stronger, Tate quotes another theorist - one Abercrombie - who 'observes' that:

> 'The mathematician argues certain conclusions from certain relations of quantity and space, which are ascertained with absolute precision; and these premises are so clear, and so free from extraneous matter, that their truth is obvious, or is ascertained without difficulty. By being conversant with truths of this nature, he does not learn that kind of caution and severe examination, which are required in other sciences, for enabling us to judge whether the statements on which we proceed are true, and whether they include the whole truth which ought to enter into the investigation. He thus acquires a habit of too great facility in the admission of data or premises, which is the part of every investigation which the physical or mental enquirer scrutinizes with the most anxious care, - and too great confidence in the mere force of reasoning, without adequate attention to the previous process of investigation on which all reasoning must be founded. It has been, accordingly, remarked by Mr. Stewart, and other accurate observers of intellectual character, that mathematicians are apt to be credulous, in regard both to opinions and to matters of testimony; while on the other hand, persons, who are chiefly conversant with uncertain sciences, acquire a kind of scepticism in regard to statements, which is apt to lead them into the opposite error.' (p.91)

Clearly, some generalisation of reasoning must follow from Tate's account, but this seems to be, as I have already suggested, simply that one is disposed to seek and supply reasons in other contexts as a result of experience of doing so in the simple context of mathematics. Unfortunately, whilst Tate gives a wealth of examples of reasoning in several different categories, he makes hardly any reference at all to mathematics. The examples he does give are examples of *inferences* which can be made on the strength of general statements. Some of these do clearly show that, for the inference in question, backing and warrant for the conclusion are based upon *authority*. For example the truth of the premise of the following is not self-evident:

> Misery, disease, and death always follow drunkenness, dissipation, and all such crimes: then vice and misery are inseparably connected. (p.96)

With this kind of example in mind we can construct our own mathematical one which does appear to rely upon more accessible evidence as its backing:

> Any number whose last digit is a zero will be divisible by 5: 166340 is divisible by 5.

But when we compare other examples of Tate's, with a mathematical one of our own making, then things are not quite so convincing. Compare Tate's:

> All animals with four feet are called quadrupeds: then, a cow must be a quadruped. A fowl is not a quadruped – why? (p.93)

with:

> All quadrilaterals which have four right angles are called rectangles: then a square must be a rectangle.

Clearly, the backing for the conclusion 'a cow must be a quadruped' is based on the authoritative statement about what is *meant* by 'quadruped'. But the mathematical argument is also based on the 'authority' of the use of words such as 'quadrilateral' in precisely the same way.

As I have said Tate gives many examples of inferences some of which are clearly based on authority or experience beyond the scope of the early experiences of a pupil. Nevertheless, some authority of naming exists in mathematics. So the strength of Tate's view still rests upon the extent to which the authority is less marked in mathematical evidence than it is elsewhere. This is something which I shall address in the following sections, as I bring other writers from the past and present into the discussion to show how they, too, have been impressed by the minimal extent to which authority operates in mathematical contexts.

However, before moving on to a closer look at how other writers have taken up the issue of authority in learning, and related issues, it is worth asking whether other activities could not provide the same purpose that Tate believed was best provided by mathematics. In particular, certain games like chess may similarly exemplify the same principles, except perhaps that one is not normally called upon to *give* reasons for chess moves. It is this in respect, where discourse is essential to the activity, that mathematics does seem to have a special place which is fairly resistant to objections and alternatives. Everyday life however can provide opportunities for asking 'why' and eliciting 'because…'. Moreover, whilst much of the evidence appealed to in everyday life is arguably based on authority, some reasoning is surely based upon backing which depends upon fairly straightforward perceptual evidence. For example such reasoning as: 'why' did you take off your jumper?' 'Because I'm hot' seems to be based on evidence which is perfectly clear. Thus to encourage reasoning of this kind would issue in a training which would meet some of the demands of Tate's theory. But it seems to me that he could still argue that mathematics provides a simple *system* of results which are missing in everyday perception and in which more sustained explanation may be sought on behalf of the pupil. But the plausibility of Tate's account still depends upon a sensitive trade-off between those aspects of his theory which distinguish it from the ambitious claims of earlier procedural accounts, and those which distinguish it from the value of reason in familiar contexts like games and everyday life.

If the modest theory of mental training based on Tate's view can be upheld, then it does seem to fulfil the demands of the HMI booklet mentioned above (see page 131), from which some important points were raised. In particular not only does such mental training not imply drill, but some intelligence is necessarily involved. The training on this theory has, arguably, fairly immediate effect too. Also whilst I have agreed that this training might not be exclusive to mathematics (since reasoning in more familiar contexts has an important part to play, the drawing of inferences, for example) mathematics does appear to be a particularly suitable candidate.

Mathematics and morality

Tate is surely right to restrict the scope of mathematical reasoning, and not to suppose that it might be applied to other areas of life in quite the way that some might suppose. There is, however, another reason why the subject might be associated with areas outside its own particular domain. Mathematics has traditionally enjoyed a certainty not possible in other subjects, even though many now believe that this is because mathematics is a formal subject devoid of meaning. Thus, when we say of some mathematics that it is right (or wrong), we are often only suggesting that a method has been used correctly (or incorrectly). In other words 'that is right (wrong)' means not so much that you have revealed a truth (or falsity) but rather that the conclusion follows correctly (incorrectly) from what is given. Whatever the source of the use of terms like 'right' and 'wrong', as ways of speaking about mathematics, the fact that they are used in this way has led people to suppose that there is a connection between mathematics and morality. Once a mathematical system is constructed, what follows from the system is inescapable. To face up to this kind of inevitability, has sometimes been likened to possessing a kind of courage. It is as though one must accept the consequences of a mathematical deduction, in the same way as one must accept the consequences of one's actions. Mathematics has thus been associated with character building or even a form of *moral* training. Myers (in Moritz, 1914 pp 69-70) for example writes:

> I do not maintain that the *chief* value of the study of arithmetic consists in the lessons of morality that arise from this study. I claim only that, to be impressed from day to day, that there is something *that is right* as an answer to the questions with which one is *able* to grapple, and that there is the wrong answer - that there are ways in which the right answer can be established as right, that these ways automatically reject error and slovenliness, and that the learner is able himself to manipulate these ways and to arrive at the establishment of the true as opposed to the untrue, this relentless hewing *to* the line and stopping *at* the line, must colour distinctly the thought life of the pupil with more than a tinge of morality ...To be neighbourly with truth, to feel one's self somewhat facile in ways of recognising and establishing what is right, what is correct, to find the wrong persistently and unfailingly rejected as of no value, to feel that one can apply these ways for himself, that one can think and work independently, have a real, a positive, and a purifying effect upon moral character. They are the quiet, steady undertones of the work that appeal to the learner for the sanction of his best judgement, and these are the really significant matters in school work. It is not the noise and bluster, not even the dramatics or the polemics from the teacher's desk' that abide longest and leave the deepest and stablest imprint upon character. It is these still, small voices that speak unmistakably for the right and against the wrong and the erroneous that really form human character. When the school subjects are arranged on the basis of the degree to which they contribute to the moral upbuilding of human character good arithmetic will be well up the list. (See Moritz, 1914 p.70)

Myer's extract seems to be related to Tate's view that mathematics is a form of exemplification. But he goes further than Tate and seems to me to be overstating the case. There are clearly two senses of 'right' and 'wrong'. Both might be regarded as normative but not both moral. To have the 'right idea' is not to suggest that one is any more moral than one was before one had such an idea. It simply means that one is following good practice. But 'good' here, need not necessarily be the moral sense of 'good'. Indeed, mathematics is arguably one of the most morally neutral disciplines, so the use of right and wrong in mathematics does not entail that one becomes any more moral by getting things right.

Mathematics and social value

However, an important point does arise here. If wrong or right mathematics were dependent upon good or bad practice, where does this practice originate? Is it the case that *authority* of some kind must enter the picture and if so where does this originate? We may acknowledge that 'right' and 'wrong' are neutral terms yet still want to insist that there is important *social* value in learning mathematics. There is a sense in which Tate has already anticipated this point of view in his suggestion that mathematics provides a forum for asking and giving reasons, and moreover that it is accessible in a way that other subjects are not. Several writers more recently have made essentially the same point though often they have invoked the social notion of 'authority' to make their point. In mathematics, they argue that what is right or wrong, true or false, can in a sense be determined by *anyone*. Sometimes this point has even been used to link mathematics and 'democracy'.

If such views can be sustained then we might have found a justification for learning mathematics that avoids the dilemma that concerned John White which I outlined earlier. Recall (see page 7) that White argued that a National Curriculum must neither advocate a particular lifestyle, nor remain neutral since in being neutral it was tacitly committed to the ideal of autonomy. Nevertheless, White points out that the 'general argument for a National Curriculum rests on the notion of democracy', which as he points out 'has to be taken as a system of government in which every citizen has an equal right to participate in the exercise and control of political power'. Aims for a National Curriculum must therefore embody this assumption. If it could be shown that mathematics provides an individual with certain dispositions required for taking his or her place in a democratic society, then this would provide some justification for including it within a National Curriculum.[123]

We can contrast a democratic society with one in which control is by the few and the individual's opinion counts for little or nothing. Decisions are based on the few people who carry *authority*. In the field of education it appears that some subjects embody authority in a similar way and are thus less democratic than others. Sawyer remarks:

> If...a French teacher tells me that the French always use one phrase to convey a particular idea, and never use the phrase that I have put in my homework, I can only bow to superior knowledge. If I produce a dictionary in support of my version, I will be told the dictionary is out of date; people do not speak like that today. (Sawyer, 1964 pp.81-2)

Often, in response to the question why the French express themselves in a certain way, the best that can be said in reply is 'because they do'. Mathematics, it seems, is different in this respect. As Skemp (1971, p110) points out: 'In mathematics perhaps more than any other subject the learning process depends on agreement, and this agreement rests upon pure reason'. Skemp argues that in mathematics the teacher must submit to exactly the same rules as the pupils so that within the sphere of mathematics authoritarian pronouncements are out of place. In particular he writes:

> If a teacher makes a mistake when working on the blackboard and a member of the class points it out, the teacher has no alternative but to correct it. *Teachers are subject to the same rules as pupils, and these are not the rules of an authoritarian hierarchy but of a shared structure of concepts.*[124] (p.110)

Fitch makes the same point when he remarks that mathematics:

> ...is the one department of school-study in which the sceptical and inquisitive spirit has the most legitimate scope; *in which authority goes for nothing.* In other departments of instruction you have a right to ask for the scholar's confidence, and to expect many things to be received on your testimony with the understanding that they will be verified afterwards.[125] (Fitch, 1902 p.320

Since it is true that authority *in some sense* is out of place in mathematics, it seems to be one small step forward for Sawyer to conclude that:

---

[123] Clearly, the practical utility of mathematics already provides some knowledge which is indispensable for the individual to take up his or her place in a democratic state. But this bears little upon character.

> *Mathematics is, or should be, the most democratic subject in the world*; in mathematics there is no evidence available to the teacher that is not available to the student.[126] (Sawyer, 1964 p.82)

But even if the gradual move from certain observations of the procedures of mathematics leads us to the view that mathematics is 'the most democratic subject in the world', that does not make the subject self-justifying in a National Curriculum. The use of democratic is not a political notion in mathematics. It does not connect with the idea of political control, though it does suggest that the manner in which mathematics proceeds has something in common with the manner in which social or political decisions are reached in a democracy. But is this true? Surely, the truths of mathematics are not straightforwardly reached by an individual's decision, even if it is true that a choice of method might be available. Surely, once a mathematical system is constructed, what follows from the system is inescapable. In this sense at least facing the inescapable in mathematics is character building.

The Transparency of Mathematics

What we must be careful about here is linking those authors, who want to rule out the authority of the teacher in settling mathematical decisions, with a questionable view of mathematics. To see what I mean by this cautious remark, consider firstly how the following extract from Augustine can be linked with the comments of the modern writers whom we have just discussed. In St. Augustine's Free-Will ii 20-24 Augustine asks his interlocutor Evodius whether he can 'find something which all reasoning beings see in common, each with his own reason and his mind' The latter replies:

> I see that there are many such things, but it will be enough to mention one of them. *The principle and truth of number is open to all reasonable beings*, in such a way that every calculator may try to grasp it with his own reason and intelligence. One man can do it easily, another only with difficulty, while another cannot do it at all. But *the truth of number is the same for all who have the power to grasp it*; it is not changed and transmuted, so to speak, into nourishment for the person who takes it. Therefore it is not truth which fails when someone makes a mistake in calculation; truth remains true and complete, and the less of it a man sees, the more he is in error. (See Howie, 1969a pp. 245-6)

---

[124] My italics
[125] My italics
[126] My italics

Augustine is one of the earliest writers who not only believes that mathematics is grasped with one's own reason rather than by the authority of others, but who also goes a little further and suggests that it is a moral shortcoming if one refuses to see the truth.[127] The tradition from St. Augustine, Tate, Skemp and Sawyer, all of whom are trying to suggest that mathematics in one way or other is either simple, free of authority, open to reason etc, is nevertheless open to one objection which needs to be considered. There is a risk that our faith in the transparency of mathematics contains some aspect of what Popper (1963) calls the 'truth is manifest' doctrine. This doctrine, which according to Popper is especially associated with Bacon and Descartes, embodies the belief that 'truth may perhaps be veiled. But it may reveal itself'. It represented what Popper calls an 'optimistic epistemology'. Its exponents, he writes:

> ... taught that there was no need for any man to appeal to authority in matters of truth because each man carried the sources of knowledge in himself; either in his power of sense-perception which he may use for careful observation of nature, or in his power of intellectual intuition which he may use to distinguish truth from falsehood by refusing any idea which is not clearly and distinctly perceived by the intellect.[128] (Popper, 1963 p.5)

An 'optimistic epistemology' is welcomed as a contrast to epistemological pessimism which Popper says is:

> ...linked, historically, with a doctrine of human depravity, and it tends to lead to the demand for the establishment of powerful traditions and the entrenchment of a powerful authority which would save man from his folly and his wickedness. (p.6)

But then he asks 'how can we ever fall into error if truth is manifest?' His reply is:

> ..through our sinful refusal to see the manifest truth; or because out minds harbour prejudices inculcated by education and tradition, or other evil influences which have perverted our originally pure and innocent minds. (p.7)

Indeed in St. Augustine we an excellent example of this move:

---

[127] Consider the old story about the mathematics teacher who when presenting some proof to his pupils announced that it was surely "obvious". However, he paused, left the room and returned after some while and repeated "Yes, it is obvious"!

[128] In recent education, too, especially amongst teachers of young children, the expression 'mathematics is all around us' has traces of the 'truth is manifest' view. After all it was supposed to show that mathematics is not so elusive after all. Yet the mathematics is only around us after we know what it is. A better expression is that *opportunities for learning mathematics* are all around us. Pi is not really in the sky!

> ...everyone who discusses the question, whom God has endowed with good ability and who is not clouded by obstinacy, is compelled to admit that the law and truth of numbers do not depend on physical sensation, but stand inviolate and pure, and are open to the common inspection of all rational beings. (See Howie, 1969a p.249)

Despite its obvious attraction, then, Popper rejects the truth is manifest view because he believed that it *did* ultimately appeal to an authority, namely, *'the authority of the senses'* and the '*authority of the intellect*'. In so doing it dogmatically refused to acknowledge human fallibility which is central to Popper's own theory. He believed that we do err, but that we get closer to the truth by acknowledging this fact and by '...persistently searching for our errors: by indefatigable rational criticism, and self-criticism'.[129] (p.16) Popper's views are important in showing that we ought not to press too far the view upon which the modest view of mental training rests.

It is perfectly reasonable to draw out those features of mathematics which show that it has *some* autonomy and does not seem to depend upon custom and tradition to the extent that other disciplines do. But to over-emphasise these features is counter-productive. Recall that the modest theory was supposed to provide an initiation into the rational enterprises. But if the very feature of this mental training – the ease by which mathematics can exemplify the giving of reasons – arises from a wholehearted belief in the transparency of the discipline, this undermines the very critical spirit which it is supposed to engender, and then of course it is self-defeating.

---

[129] In mathematics Popper's approach was developed by Lakatos (1976).

Summary and Conclusion of Part 3

A distinction between the so-called *practical* utility of mathematics and its purpose of mental training may I think be drawn out by distinguishing between those by whom mathematics is used and the kind of ends which result. When mathematics is consciously used by an individual as a *tool* to get something done, which is fairly clearly defined and perhaps relatively immediate, then mathematics may be said to have practical utility. But mathematics may also be regarded – 'used' if you will - as what I have called a '*vehicle*'. In this respect an individual might simply display his or her virtuosity but, additionally, an educator might use mathematics as a vehicle not simply to display mental acumen in his pupils, but more importantly to improve the mind in some *extra-mathematical* way.

It might be supposed that all mental training is simply a particular capacity to carry out an operation of some kind. But I have shown that in the area of *health*, or more specifically stress therapy, we do have a form of mental training, namely, meditation, which is not strictly undertaken in order to facilitate some determinate activity. It is supposed to promote a more indeterminate outcome of personal well-being. This provides us with one clear model of the outcome of mental training. But what needs to be shown is how the *learning of mathematics* could hope to produce personal well-being of a kind like this. I have tried to show that certain theories of mental training do suppose that this end may be achieved, through the study of mathematics, especially if we regard this end as *self-mastery*.

I have outlined *three* different models of mental training by tracing back those whom I see as their earliest exponents and also by following through their legacies. The three accounts derive from Plato, Descartes and Tate. Since each of them is connected in some way to the acquisition or the development of *reason* and that this has been regarded as essential to promoting self mastery, I have connected these theories, to a lesser or greater extent, to this educational ideal. The nature of reason, which each of these main theorists believe mathematics has a part in achieving, has changed throughout history. With Plato it seems to be an object of contemplation, with Descartes and Fitch certain kinds of procedure. Indeed, with Fitch it has become identical with *logic*. Tate however is concerned less with the *source* of reason *per se* and more with the *practice* of reason based on 'mathematical evidence'.

We have seen that Plato was curiously preoccupied with the special nature of mathematics to produce 'paradoxes', in contemplation of which we find ourselves engaged in philosophy and drawn closer towards reified Reason. Reason thus conceived as the Form of the Good, becomes the ultimate object of contemplation which reveals what it true, good and beautiful, and why it is so. The change in metaphysical background makes Plato's theory difficult to accept *in toto*. Nevertheless, I have argued that his paradoxes have remained alive throughout the centuries. For this reason they ought not to be ignored. But more that this, I have shown that at least one recent theorist, Brent Davis, has developed a theory which whilst not owing anything consciously to Plato is still partly based on the revelation which the 'unsayable' in mathematics is supposed to provide.

The most familiar theory of mathematics as a form of mental training is probably that which derives from Descartes. This theory identifies reason not with something existing independently and 'outside' the individual, as it was for Plato, but with certain *procedures*. The rigorous *deductive* method based on *intuitively* known premises and rules of passage, by which mathematical results are validated, if not originated, are supposed on this theory to have application in extra-mathematical contexts. Indeed they are supposed to be the universal canons of truth acquisition. It is no surprise that mathematics is thus seen as the subject *par excellence* for training the mind. With Descartes such training still leads to *self-mastery* in the sense that awareness of the centrality of procedures allows an instrumentalisation of the passions, which have now become separated from the mind, to take place.

The attraction of what is essentially Descartes's view still persists at the end of the nineteenth century when it is restated by Fitch. By then mental training has become training in logic, again of universal application. In a sense, then, we may say that one of the predominant overarching views of mental training through mathematics arises through the connection of the discipline with *reason*. But reason has in a sense become transformed, perhaps even emasculated, as it has changed from being part of the furniture of the universe, to the procedures carried out by rational individuals to the application of logic. To this extent alone the justification of mathematics for this purpose has also lost some of its conviction. But there are other reasons why the Descartes/Fitch procedural view of the matter cannot in the end carry much, if any, weight in educational justification. This is because by no means *all* of the areas of knowledge may be seen to be so tightly articulated by principles which the procedures of mathematics may serve. This is clearly shown by Passmore and Hamlyn. Moreover, as Toulmin has demonstrated, in general, the principles of argumentation have fewer invariants to share with mathematical arguments than we might suppose.

It remains for Tate and his legacy to provide us with a similar but more modest theory of mental training yet still identified as training in *reason*. What he believed was the *simplicity* of mathematical *evidence* makes it a suitable backing and warrant (in Toulmin's sense) for inference making and thus an ideal subject for initiating pupils into rational discourse where the business of giving evidence is necessary. But the scope of mathematical evidence is not extended into all areas of life. In this respect it is distinguishable from the other two broad mental training views. It is thus at its best in the early stages of education and is only generalisable with the respect that pupils are disposed to seek and supply reasons *of some kind* in mathematics and elsewhere.

Tate was particularly impressed by the fact that mathematics is less dependent upon authority than other areas of life. Thus when a pupil is invited to give reasons, those which he or she gives are somehow more *authentic*. If Tate is correct in this respect then the mathematical forum is distinguishable on the one hand from the rule following activities like chess, where reasons for moves are available but not normally articulated, and from ordinary discourse where reasons are sought but which still might need to depend upon authority.

Others, too, have continued in Tate's tradition. The inappropriateness of authority, even the democratic nature of mathematics has for some writers seemed to give the subject an attractive social dimension. At its most extreme we can characterise the Tate legacy as adopting something like the 'truth is manifest' view which Popper has famously identified and duly criticised. For all that, it seems to me that Tate's view provides us with the best model for a mental training which meets the suggestion of the kind referred to in the HMI booklet. Clearly, some reasons in everyday experience are based on direct perception and thus, like mathematics, do have simplicity and certainty. Such reasoning opportunities mathematics cannot hope to displace. Nevertheless, mathematical evidence is, additionally, of a somewhat broader kind and this provides formal and rich opportunities, especially for young pupils, to make inferences.

Summary and Conclusion

Now that the discussion is finished it is time to draw the parts together and see what the overall conclusions are, what shortcomings it may have and how further work might contribute to the lines of enquiry which I have begun. The value of the task, as I suggested in the introduction, lies in showing how mathematics contributes not only to the general question of the *aims of education* but also, to some extent, to the question of *cognition*. Moreover, whilst several writers have already approached the topic of why we learn mathematics and what its aims should be, it has been my contention that these are not especially illuminating. One reason for this is that they have relied rather too much on one or other of two approaches; I will elaborate on these a little further and say how my enquiry stands in relation to them.

In seeking a rationale for learning mathematics one approach is to examine the *nature* of mathematics, to consider certain of its *properties*, and thus to draw out aspects of mathematics which either seem to be obviously educationally worthwhile, or which may be shown to have some significant educational worth by further argument. For example, we may be struck by the fact that mathematics is a kind of tool for practical purposes, a system of logical relations, a powerful language, a source of fascination etc. Then, if we believe that education should be concerned with empowering pupils by providing them with tools, developing logical thinking, acquiring language, stimulating the imagination etc, then the value of mathematics is confirmed and we have some reasons for teaching it.

But the nature of mathematics is not as straightforward as it may seem to be. There is room for disagreement upon precisely what it is. I have taken a fairly orthodox view of mathematics, in accordance with most of the theorists whose work I have explored. I have not, for example, explored the so-called 'enactivist theory' which Brent Davis partly bases his conclusions upon. This is because I was more concerned to draw out the similarities between his and Plato's theory. Moreover, although my discussion finishes with some of Popper's views I have not explored the 'fallibilist'[130] theory of mathematics, which grows out of his philosophy. On this theory the nature of mathematics is not simply the systematic body of knowledge derived from axioms and characterised by certainty, which I have supposed it to be. However, whilst alternative

[130] See for example Ernest (1994) and Lakatos (1976)

views on the nature of mathematics may open up other avenues, they are just as likely to confirm some of my conclusions. I have found powerful arguments to show that mathematics is not a fine art, but if mathematics is regarded more as a *critical process* than a *product* this would appear to provide additional support for my arguments. It is less clear whether my critique of mental training would be undermined by fallibism. Whilst the supposed *certainty* of mathematics impressed each of the theorists I have discussed, the importance of processes was just as important to Descartes, Fitch and Tate. Nevertheless, it must be borne in mind that the fitness of purposes which I have been seeking must be seen in terms of a particular conception of mathematics.

Another way of approaching the question, of seeking a rationale for mathematics education, is to look at things the other way round. Then what is first attempted is a deliberate spelling out of those aspects of education which are of special value: creativity, autonomy, aesthetic experience, self-sufficiency, collaboration, the awareness of other cultures etc. It is then argued that mathematics is at least one way in which some of these ideals may be achieved, and once again the value of mathematics is supposed to be confirmed. I have approached my enquiry from *some* educational ideals. In particular, I have approached the question from an aspect of personal autonomy - *self-mastery* - and also from the goal of *self-knowledge*. But other goals remain which may be linked to mathematics, particularly though not exclusively, that of *citizenship* which I have not mentioned explicitly, though I have given some consideration to *democracy*. The question of the extent to which mathematics can contribute to citizenship is one for which further detailed work would be fruitful.

But to say that mathematics is *one* way in which certain educational goals are attained is not necessarily to show that it is the *best* way. My research has shown that across the various aspects that I have explored, there is disagreement amongst the remarks of Plato, Locke, Leibeck, Whitehead, Stolnitz, Huntley, Poincaré, Papert, Fitch, Watson, Tate and St. Augustine about the accessibility of some of the valued aspects of mathematics. These writers may of course have different levels of mathematics in mind. Nevertheless, the fact that they take issue at all means that we cannot simply take it for granted that if certain properties exist in mathematics then they exist for all as a rationale for learning the subject. We must take this point seriously. In particular, the amount of time taken to be in a position to appreciate the aesthetic properties of mathematics might not make mathematics the *best* way of achieving such goals. Further empirical work of the kind carried out by Papert (1993) and of that discussed by Dreyfus & Eisenberg (1986) is necessary to confirm that what exists is sufficiently accessible.

In practice, these two approaches to the question of seeking a rationale for mathematics are not so clear-cut. They are no more clearly cut than the way in which a mathematical proof is obtained by starting from premises and working one's way to the conclusion. The psychological processes may have indeed involved working forwards from the premises to the conclusion, or backwards from the conclusion to the premises, but it is just as likely that a combination of both are at work. Similarly, we may begin with some educational ideals and work towards the nature and properties of mathematics and hope that any gaps that appear on the way can be filled. But there is always the risk that in determining what we believe to be the nature and properties of mathematics we are covertly looking for something which we already value. In examining education, too, we may already be selecting those goals which we believe align themselves to particular aspects of mathematics. There is a narrow line between setting forth a rationale and making a rationalisation.

My research has revealed how important it is to follow through carefully *both* the connections from mathematics to educational goals *and* those from educational goals to mathematics. This is especially true of the section on mathematics and art in Part 2. Little or no attempt is made by certain prominent theorists, most of whom are mathematicians, to trace back the educational implications of asserting that mathematics ought to be regarded as a fine art. Of course, as mathematicians, those writers were not intending to make explicit educational connections between mathematics and its resemblance to the fine arts. But it is important to note that a rationale for mathematics cannot be derived simply from connecting mathematics and fine art. To adopt this approach leaves the crucial connection between fine art and education not spelt out. In such cases what we have is simply what I have called 'justification by association'. The consequence of this is clear: the value of mathematics still depends upon an adequate rationale for that with which it is supposed to be associated, in this case the fine arts. So rather than providing an answer it simply relocates the question.

Most of those who have only provided justification by associating mathematics with the fine arts have constantly made reference to the same two broad aspects of mathematics:

the creativity involved in its origination and certain aesthetic features, typically beauty, which it is supposed to have. Additional attempts to deal with the absence of intrinsic emotion in mathematics, by trying to show that the presence of emotion in the fine arts is less characteristic than we might ordinarily suppose, I have argued, is not convincing either. Of course, anything may be *indexed* as fine art by the 'artworld'. But this only shows that it is time to look at the matter the other way round and take the alternative approach; we need to consider what educational ideals would be addressed by indexing mathematics in this way.

In considering the special way in which the fine arts can contribute towards the particular educational goal of *self-knowledge*, outlined by White, I have argued that mathematics could never equal them in this respect. The particular kind of *self-knowledge* which the fine arts can provide depends upon treating *human perception* as an end, and not simply as a starting point. Mathematics, like Science, must at some point leave the human point of view behind and in this important respect can not be considered to be a fine art. But it has to be emphasised that the examination of mathematics alone does not lead to this conclusion. Careful consideration must be made of the fine arts and their role in education. Yet this referral back to art and education does not produce an altogether negative conclusion. In considering *mathematics*, once again, one writer was found who believed that self-knowledge, of a kind, *is* realisable through mathematics and thus that mathematics is a fine art. As it turns out, the kind of self-knowledge obtained from mathematics is of a narrower, less pressing kind. Furthermore since such self-knowledge arises largely from a consideration of how mathematical systems are *created* rather than discovered, this view has the same defects as others which invoke creativity. More importantly its goal is of dubious accessibility in general education.

Taking our attention away from mathematics and towards art facilitates further fruitful analysis. The attempts to assimilate mathematics to the fine arts are extravagant but they are not wholly in vain. By disentangling the usual connections between fine art and aesthetics – by showing that art and aesthetics are not coexistent - a more defensible claim can be upheld, namely, that aesthetic properties in mathematics exist. Such properties can be linked straightforwardly to the educational ideal of personal enrichment through the enjoyment achieved by either contemplating or discursively attending to objects in a non-purposeful way. Although I have taken one aesthetician's notion of *pattern* as my main point of entry into mathematics, I have shown that aesthetic experience is available, to some extent, at different levels. At one level such experience is possible by *contemplating* or *following* mathematical pattern or argument and appreciating it simply for its formal properties. At another level what is involved is *producing* mathematical argument which meets certain formal ideals. At a higher level, though scarcely accessible in school, similar aesthetic ideals may be met in the studying or *constructing of mathematical systems*. In addition, I have also pointed out that not all reference to aesthetics in mathematics links directly to the non-purposeful. Aesthetics in mathematics is not always a matter of *experience*. It has been suggested that the unconscious working of the mind is regulated by an aesthetic sense, more developed in some individuals than others, which is *functional* in selecting appropriate mathematical solutions to problems. In a similar way, the pursuit of mathematical games and puzzles, purely for the fascination which engagement in them brings, constitutes an important purpose of mathematics. But the use of games as a motivational factor is quite compatible with a thoroughgoing justification of mathematics on the strength of its practical utility. It is on this very point that the distinction between aims and purposes becomes important. Aesthetic appreciation can be something we aim at without necessarily being what mathematics is partly *for*.

If several of the arguments I have discussed surrounding art and aesthetics, began from a consideration of the nature of mathematics, the case for mental training began from the other end. In using meditation as a very simple model of achieving a kind of mental composure in stress relief, I sought in mathematics the wherewithal to bring a comparable kind of well-being. This led me to one of the earliest coherent theories of education and one which did begin by outlining personal and societal well-being before going on to attempt to show how mathematics might provide some of this. Thus Plato seeking the Just State realised that this is a function of a just individual who is so, only when his potentially competing passions are in harmony. He therefore sought a means of providing self-mastery. Since such harmony is ultimately achieved by contemplating the Form of the Good, or in other words by being ruled by reason, then anything which sets one on this road is going to be potentially educationally worthwhile. A consideration of mathematics shows that it sits midway on the road and hence acts as a stepping stone on the ascent to the Good. Its puzzling properties, too, stimulate the mind and are supposed to shake the individual out of his complacency. I have also shown that the cognitive use of mathematics if it ultimately calms the mind, is unlike the fairly mechanical process of meditation. It is first necessary to rouse the mind into a fairly long and arduous task of finding reason. For all this, Plato still does not prove a particularly convincing case. His consideration of the nature of mathematics as something which provides paradoxes, whilst being of some philosophical interest, is scarcely sufficient to do the job he thinks it can, especially if we strip off the metaphysical background which is implied in his theory.

Descartes, too, begins away from mathematics, as it were, but barely in the area of education, and seeks a united approach to the pursuit of the sciences. In so doing he shows how mathematical procedures should be the model upon which to construct any enterprise in which truth is sought. Such general procedures which mathematics is supposed to provide persist in later writers like Fitch. I have argued that the generalisability of such processes is not tenable. The most plausible theory of mental training I have argued is one in which mathematics provides clear and simple evidence *for* initiation into inference making, not as a set of principles *of* reasoning which may be applied across other areas of discourse.

My enquiry has been centred round an exploration of the question of the purposes of mathematics education. The two questions 'why do X?' and 'what's the use of X?' though not identical in meaning, are nonetheless somewhat interchangeable in uncritical discourse. We are so prone to equating the reasons for doing something with the use that it has, that the questions 'why do mathematics?' and 'what's the use of mathematics?' have in education, as elsewhere, a close similarity in meaning. I hope that it is clear that my enquiry has shown that they *not* equivalent.

In a trivial sense 'why do X?' and 'what's the use of X?' are not equivalent simply because, as I have shown, theorists since ancient times have set out reasons for doing mathematics which are contrasted with at least one particular idea of usefulness. That is to say, such reasons are supposed to be distinct from those which concern the application of mathematics to achieve fairly immediate and accessible extra-mathematical ends in everyday life on behalf of the possessor of mathematical knowledge. But to set out supposedly non-utility reasons does not imply that these are *good* reasons. Indeed, other theorists, perhaps by equating the very questions 'why do mathematics?' and 'what's the use of mathematics?' or perhaps working with a more generous interpretation of the notion of 'useful', characterise *all* reasons for learning the subject in terms of use. I have shown that beyond even a fairly broad notion of utility there are non-trivial reasons for learning mathematics. Doing mathematics for its own sake, though a more complex notion than it appears, *is* a possibility. In one of its interpretations, which involves engaging in mathematics for enjoyment, mathematics is not reducible to any helpful notion of usefulness. I have argued that an educator's *use*, if it must even be called that, of mathematics as a vehicle for training the mind, is also separable from the utility of mathematics conceived as something which can be applied by the possessor.

I have been at pains, then, to identify, describe, analyse and criticise non-utility reasons for engaging in mathematics and I want to defend them against being swallowed up under the name of utility. Nevertheless I defend them *guardedly*. This is because whilst admitting that non-trivial non-utility reasons for learning mathematics do exist, they are nevertheless open to question in a way in which the utility ones are not. This is an important part of my conclusion and I will enlarge upon it presently.

Whether or not we do equate the question of the value of mathematics with its use, much, if not all, of mathematics is developed for fairly transparent practical application. Even mathematics that is not developed with a clear practical end, is nevertheless valued highly if it is useful within the discipline itself. History, too, shows us that it has been central to the growth and development of civilisation, that civilisation would have been impoverished without the development and application of mathematics. It is evident, too, that mathematics is still being used and it takes little to convince anyone that it will always be used in the future.

Nevertheless, there still, of course, remains a gap between remarking that something has uses and thereby asserting that it has educational value. One difficulty is the extent to which mathematics is supposed to be *universally* useful. What is implicit in *many* rationales for learning mathematics is not only how *some* individuals, but rather how *all* individuals, will make use of the mathematics prescribed for them. My section on counting quite clearly shows that this particular aspect was developed for its utility and remains of paramount importance to all. It is impossible to gainsay this, though it is of importance to ensure that pupils are aware of it. But it is often not this aspect of mathematics, nor even those other areas of rudimentary mathematics, some of which I have also reviewed, that are in question when the reasons for learning mathematics are raised. It is the justification of the sustained study of the more complex areas which is often sought. I have in mind those areas of mathematics which many find very difficult to master and for which considerable time and energy is required, time and energy which could be used for other things thought to be more worthwhile. So, although the utility of mathematics is at once self-evident, its force as a justification for *all* pupils in school may nevertheless be inversely proportional to the depth and breadth of the content under consideration.

Of course doubts about the extent to which utility is self-sufficient as a justification, beyond the rudimentary stages, may be unfounded. Further work on the universal value of mathematics for such areas as citizenship, for example, might throw light on this problem. Nevertheless, it is when a particular level of mathematics seems to be hard to justify, for all individuals at least, that non-utility reasons are sometimes introduced to supplement, or even more ambitiously, to replace the utility reasons.[131] When doubts are raised about the amount of time and effort involved in learning the subject, these other non-utility reasons are sometimes set out to augment the rationale. It is has therefore been of crucial importance to examine carefully the non-utility reasons to see how fit they are for mathematics education. If the non-utility reasons are of only marginal importance, then the utility arguments are left to stand up almost unaided.

I come now to my guarded comments about non-utility reasons for learning mathematics. If at least one of the aims of exploring non-utility reasons is to see whether the self-evident utility reasons, which are convincing in the rudimentary aspects of mathematics, may be supplemented by other reasons, then my work has shown that difficulties emerge. Firstly, the most plausible view of mental training stresses how it is particularly the procedures within the *elementary* stages of mathematics that are appropriate for such training. Indeed if this 'modest' view of mental training is to have value it is valued precisely because it provides an *early* initiation into rational enterprises. If we delay this process then the other disciplines can take care of themselves. If this is so then a justification does arise, but at a place in mathematics where a supplementation of reasons is scarcely necessary, where the utility rationale is arguably at its strongest. The same point seems to be true even in a less plausible view of mental training where mathematics is supposed to provide stimulation for reflective thinking. For Plato such reflection depended upon a mature approach to the subject, yet it is in the elementary areas of mathematics that this typically takes place. So that to benefit from the paradoxes or the 'unsayable' in mathematics few convincing reasons are given by Plato, or recently by Davis, for sustained learning in complex areas of mathematics for this purpose, even if such areas are not ruled out as appropriate 'sites' for contemplation and self-knowledge.

But if the case for mental training does not provide convincing support for utility reasons as the subject matter goes beyond the rudimentary, it may be supposed that the case for aesthetics does. But my earlier remarks do show that if accessibility to aesthetics is a difficulty then this might be just the property which comes into play in further study of mathematics, if not earlier. So this is a possible candidate for supplementing the utility reasons where, as I am suggesting, they are needed most. Nevertheless, the point which I made earlier now comes in to undermine the situation: several writers invoke aesthetics as a means to learning mathematics for its *utility*. Aesthetics becomes a means of ensuring that the learning is successful. This is what I meant by saying that the aesthetic reason is 'unstable'. Either way my work shows that the non-utility views are not of much help in supplementing the utility views where the latter might seem least convincing.

But as I have already remarked, further work on the extent to which the utility of mathematics beyond a rudimentary stage is sufficient for a rationale still needs to be undertaken. Without this, however, we still have the entertainment value of mathematics as a possible justification for mathematics at all stages and the pursuit of aesthetics in at least the later stages.

Even if my work has not revealed reasons which can replace or supplement the usual utility reasons to any considerable extent, it has been important in another way. It has shown by a careful, if selective, examination that mathematics does have some additional value even if this is not always appropriate for all pupils at all levels. Nevertheless, a sensitive teacher who is aware of this value will still in many cases be able to communicate it to his or her pupils at least some of the time.

---

[131] See in particular Whitcombe (1988) and Andrews (1998) who are both explicitly sceptical about the utility justification.

Bibliography

Andrews P. (1998) Peddling the Myth, *Mathematics in School*, **27**, 2

Aristotle (1953 edn) *Ethics*, Harmondsworth: Penguin.

Aristotle (1998 edn) *The Metaphysics*, Harmondsworth: Penguin.

Beardsley, M. C. (1958) Aesthetics: Problems in the Philosophy of Criticism, Harcourt Brace

Berkeley, G (1962 edn) *The Principles of Human Knowledge*, Glasgow: Collins.

Binkley, T (1977) Piece: Contra Aesthetics, *The Journal of Aesthetics and Art Criticism*, XXXV 265-277

Champlin, T.S. (1987) Doing Something For Its Own Sake, *Philosophy* Vol. 62 no 239.

Charlton, W (1970) *Aesthetics,* London: Hutchinson.

Cornelius, M (1982) Why teach mathematics? in *Teaching Mathematics,* Cornelius M. (ed) London: Croom Helm

Costello, John (1993) The Precious Futility of Arithmetic, *Mathematics in School* November 1993

Cockcroft et al (1982) *Mathematics counts*. London: HMSO

Davis, B (1995) Why Teach Mathematics? Mathematical Education and Enactivist Theory, *For the Learning of Mathematics* 15, 2 June 1995

Davis, P.J. and Hersh, R. (1980) *The Mathematical Experience,* Harmondsworth: Penguin

Descartes, R (1969 edn) *The Essential Descartes,* Wilson, M. D. (ed), New York: New American Library.

Dreyfus, T & Eisenberg, T (1986) On the Aesthetics of Mathematical Thought, *For the Learning of Mathematics* **6**, 1

Elliott, R.K. (1971) Versions of Creativity
Proceedings of the Philosophy of Education Society of Great Britain
Vol. V, No. 2

Elliott, R.K. (1974) Education, Love of One's Subject, and the Love of Truth
Proceedings of the Philosophy of Education Society of Great Britain
Vol. VIII. No.1 January 1974

Elliott, R.K. (1974) Education and Human Being, in
Philosophers Discuss Education Brown, S. C.(ed) London: Macmillan.

Ernest, P (1994) The Dialogical Nature of Mathematics in Ernest, P. (ed) in
Mathematics, Education and Philosophy London: The Falmer Press.

Fitch, J (1902) *Lectures on Teaching,* Cambridge: Cambridge University Press.

Frege, G (1980 edn) *The Foundations of Arithmetic* Eng. Trans. J.L. Austin Basil Blackwell

Geach, P. (1957) *Mental Acts*, London: Routledge & Kegan Paul

Griffiths, H.B. and Howson, A.G. (1974) *Mathematics Society and Curricula,* Cambridge: Cambridge University Press

Hahn, H. (1956) The Crisis in Intuition in Newman, J R (ed)
The World of Mathematics Simon Schuster New York

Hamilton, E. and Huntingdon, C. (1961) *Plato, The Collected Dialogues,*

Princeton: Princeton University Press

Hamlyn, D.W. (1967) The Logical and Psychological Aspects of Learning
in *The Concept of Education* Peters, R.S. (ed) London: Routledge & Kegan Paul

Hanfling, O (1992) The Problem of Definition
in *Philosophical Aesthetics* Hanfling, O (Ed) Oxford: Blackwell

Hardy, G.H. (1967) *A Mathematician's Apology* Cambridge: Cambridge University Press

Hirst, P (1965) Liberal Education and the Nature of Knowledge in *Philosophical Analysis and Education* Archambault, R.D. (ed) London: Routledge & Kegan Paul

HMI (1979) *Mathematics 5 to 11: A Handbook of Suggestions,* London: DES

HMI (1987) *Mathematics from 5 to 16*, London: HMSO

Hofstadter, A and Kuhns, R (1964) (ed) *Philosophies of Art and Beauty* Chicago.

Howie, G (ed) (1969a) *St. Augustine: on Education*, Chicago: Henry Regnery,

Howie, G (1969b) *Educational Theory and Practice in St. Augustine*, London: Routledge & Kegan Paul

Howson, G. (1982) *A History of Mathematics Education in England* Cambridge: Cambridge University Press

Howson, G. (1991) *National Curricula in Mathematics*, Leicester: The Mathematical Association

Huckstep, P. (1999) How Can Mathematics be Useful? *Mathematics in School* 28,1

Huntley, HE (1970) *The Divine Proportion* Dover

Klein, J (1968) *Greek Mathematical Thought and the Origin of Algebra*, New York: Dover

Kline, M (1972) *Mathematics in Western Culture* Harmondsworth: Penguin.

Kline, M (1980) *Mathematics, The Loss of Certainty*, Oxford: Oxford University Press.

Koestler, A (1959) *The Sleepwalkers*, Harmondsworth: Penguin.

Lakatos, I (1976) *Proofs and Refutations* Cambridge: Cambridge University Press

Leibeck, P (1984) *How Children Learn Mathematics*, Harmondsworth: Penguin.

Locke, J (1902 edn) *Some Thoughts concerning Education*, Cambridge: Cambridge University Press.

Malcolm, A (1730) *A New System of Arithmetick*, London.

Moore, T.W. (1982) *Philosophy of Education: An Introduction*, London: Routledge & Kegan Paul.

Moritz, R. E. (1914) *On Mathematics and Mathematicians*, Dover

Munn, P (1997) Children's Beliefs about Counting in *Teaching & Learning Early Number* Thompson I (ed) Buckingham: Open University Press.

Nagel, T. (1979) The Fragmentation of Value, in *Mortal Questions*, Cambridge: Cambridge University Press

Newman, J. R. (ed) (1956) *The World of Mathematics*, New York: Simon Schuster.

Niss, M (1994) Why do we Teach Mathematics in School? in *Mathematics Teaching Conference Report*. University of Edinburgh

O' Hear, A (1988) *The Element of Fire: Science, Art and the Human World* - Routledge

Ogilvy, S. & Anderson, J (1988) *Excursions in Number Theory*, New York: Dover

Papert, S. (1993) The Mathematical Unconscious in *Mindstorms* New York: Harper Collins.

Passmore (1970) *The Perfectibility of Man*, London: Duckworth.

Passmore, J (1980) *The Philosophy of Teaching*, London: Duckworth.

Plato (1955 edn) *The Republic*, Harmondsworth: Penguin.

Plato (1956 edn) *Protagoras and Meno*, Harmondsworth: Penguin.

Plato (1970 edn) *The Laws*, Harmondsworth: Penguin.

Poincaré, H (1956) Mathematical Creation, in Newman, J R (Ed) *The World of Mathematics* Simon Schuster New York.

Popper (1963) *Conjectures and Refutations*, London: Routledge & Kegan Paul.

Popper, K.R. (1972) *Objective Knowledge*, Oxford: Oxford University Press.

Quine, W.V. (1969) Speaking of Objects
in *Ontological Relativity & other essays*, New York: Columbia University Press.

Reynolds, D et al (1998) *The Implementation of the National Numeracy Strategy*, DfEE.

Sawyer, W.W. (1955) *Prelude to Mathematics* Harmondsworth: Penguin

Sawyer, W.W. (1964) *Vision in Elementary Mathematics* Harmondsworth: Penguin.

Schopenhauer (1819) The World as Will and Idea reprinted in Hofstadter, A and Kuhns, R (op. cit.)

Sierpinska, A (1994) *Understanding in Mathematics*, London: The Falmer Press

Skemp, R.R. (1971) *The Psychology of Learning Mathematics* Harmondsworth: Penguin

Stolnitz, J (1960) The Aesthetic Attitude in John Hospers (1969)(ed)
Introductory Readings in Aesthetics
The Free Press New York Collier-Macmillan, London

Straker, A (1986) Procedures and Algorithms in Primary Mathematics
Mathematics in School Vol. 15 No 4

Sullivan J. W. N. (1956) Mathematics as an Art in Newman, J R (Ed)
The World of Mathematics Simon Schuster New York.

Tate, T. (1854) *The Philosophy of Education*
Longman, Brown, Green and Longmans,

Taylor, C (1989) *Sources of The Self* Cambridge University Press.

Toulmin, S. (1958) *The Uses of Argument*, Cambridge: Cambridge University Press

Warner, R (1958) *The Greek Philosophers*, New York: New American Library.

Warnock, M (1976) *Imagination*, London: Faber.

Watts, I. (1809) *The Improvement of the Mind* London

Wells, D (1989) Why do Mathematics? *Mathematics Teaching* 127

Whitcombe, A (1988) Creativity Imagination Beauty, Mathematics in School March 1988

White, J (1982) *The Aims of Education Restated*, London: Routledge and Kegan Paul.

White, J (1990) *Education and the Good Life*, London: Kogan Page.

Whitehead, A. N. The Mathematical Curriculum in *The Aims of Education* Williams and Norgate.

Wringe, C (1988) *Understanding Educational Aims* London: Unwin Hyman

Velleman, D. J. (1994) *How to Prove it*, Cambridge: Cambridge University Press

Yeldham, F.A. (1936) *The Teaching of Arithmetic through Four Hundred Years*, London: George G. Harrap